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PRACTICAL APPLICATION OF FOURIER'S SERIES TO HARMONIC ANALYSIS.

By R. W. Prentiss.

In many experimental investigations in acoustics, electricity, astronomy, etc., Fourier's theorem may be frequently applied in analyzing compound harmonic curves when the coördinates of certain points on the curve are known or can be measured; and in finding the terms of different periods which enter into a periodic function when certain values of the function are given or derived from observation.

Many, however, are deterred from its use in cases where it would give valuable results by the great labor required in applying it even in a modified form. Attempts have been made from time to time to reduce this labor to a minimum by methods of approximation. But for the most part the schemes presented are not practical in a simple way, or are concealed in little-known monographs or inaccessible papers.

The object of this paper is to present a scheme or algorithm by the use of which curves involving frequencies from 1 to 5, 1 to 11 or 1 to 17 times the fundamental may be analyzed mechanically and with comparatively little labor. The use of this scheme does not require any knowledge of the theorem or consideration of the formulæ. It is altogether mechanical, and, when once mastered by following the directions and explanations given in connection with the special example at the end of this article, may be used with great facility.

At the same time it seems desirable to present the considerations

which have led to the formulation of this scheme in order to indicate the way in which it may be extended to other cases.

Fourier's theorem is usually stated mathematically in the following form :

$$(1) \qquad y = f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

or

$$y = \Sigma a_r \cos rx + \Sigma b_r \sin rx \qquad (r = 0, 1, 2 \cdots \infty).$$

It is shown in mathematical treatises on the subject that the coefficients a_r, b_r may be obtained by the following formulæ:

$$(2) \qquad \pi a_r = \int_0^{2\pi} y \cos rx\, dx, \quad \pi b_r = \int_0^{2\pi} y \sin rx\, dx.$$

The use of these formulæ, however, presupposes a knowledge of $y = f(x)$, while in the experimental investigations under consideration only a few values of y are known, and the object is to find y in order to get the component periods or frequencies.

For the present purpose it is necessary therefore to use an approximation which can be carried to the degree of accuracy required by the problem in hand.

In either formula (2) the integral may be regarded as the limit of the sum of finite quantities : $y_1 \cos rx_1 \Delta x$, $y_2 \cos rx_2 \Delta x$, etc.; so that denoting these approximate values of a_r, b_r by R, r respectively, we have according to the principles of the calculus :

$$(3) \qquad \begin{aligned} R = a_r \text{ (appr.)} &= \frac{1}{\pi}(y_0 \cos rx_0 \Delta x + y_1 \cos rx_1 \Delta x + y_2 \cos rx_2 \Delta x + \text{etc.}), \\ r = b_r \text{ (appr.)} &= \frac{1}{\pi}(y_1 \sin rx_1 \Delta x + y_2 \sin rx_2 \Delta x + \text{etc.}), \end{aligned}$$

where y_0, y_1, $y_2 \cdots$ are ordinates corresponding to abscissas x_0, x_1, $x_2 \cdots$, Δx apart ; these coördinates being given or obtained as the result of measurements made on the curve to be analyzed.

Suppose the period to be divided into $2n$ equal parts, so that we have $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2x_1 = 2\Delta x$, $x_3 = 3x_1 = 3\Delta x$, etc.

$$\Delta x = \frac{2\pi}{2n} = \frac{\pi}{n}.$$

Substituting these values in (3):

$$(4)\quad R = \frac{1}{n}(y_0 + y_1 \cos rx_1 + y_2 \cos 2rx_1 + \cdots y_{2n-1} \cos (2n-1)rx_1).$$

$$(5)\quad r = \frac{1}{n}(y_1 \sin rx_1 + y_2 \sin 2rx_1 + \cdots y_{2n-1} \sin (2n-1)rx_1),$$

or, in particular, denoting by $A, B, C \cdots$, $a, b, c \cdots$ the first, second, third, $\cdots$ coefficients of the cosine and sine terms respectively in the approximate Fourier Series:

(6)

$$nA = y_0 + y_1 \cos x + y_2 \cos 2x + y_3 \cos 3x + \cdots y_{2n-1} \cos (2n-1)x,$$

$$nB = y_0 + y_1 \cos 2x + y_2 \cos 4x + y_3 \cos 6x + \cdots y_{2n-1} \cos 2(2n-1)x,$$

$$nC = y_0 + y_1 \cos 3x + y_2 \cos 6x + y_3 \cos 9x + \cdots y_{2n-1} \cos 3(2n-1)x,$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$na = y_1 \sin x + y_2 \sin 2x + y_3 \sin 3x + \cdots y_{2n-1} \sin (2n-1)x,$$

$$nb = y_1 \sin 2x + y_2 \sin 4x + y_3 \sin 6x + \cdots y_{2n-1} \sin 2(2n-1)x,$$

$$nc = y_1 \sin 3x + y_2 \sin 6x + y_3 \sin 9x + \cdots y_{2n-1} \sin 3(2n-1)x.$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

As the further development of the general formulæ for *any* number of divisions is aside from our present purpose, the special case in which $2n = 36$ may be given to illustrate the method and to show how the scheme finally adopted is formed.

The compound curve being given (by photography, etc.) divide its period into 36 equal parts by abscissas $x_0, x_1, \cdots x_{35}$. Measure with the required accuracy the corresponding ordinates $y_0, y_1, \cdots y_{35}$; for convenience choosing the axis of x below the lowest point of the curve that these ordinates may all be positive.

In this case $2n = 36$ and $x_1 = \Delta x = 2\pi \div 36 = 360° \div 36 = 10°$ and formulæ (6) become :

$$18A = y_0 + y_1 \cos 10° + y_2 \cos 20° + \cdots y_{35} \cos 350°$$

$$18B = y_0 + y_1 \cos 20° + y_2 \cos 40° + \cdots y_{35} \cos 700°$$

$$\vdots$$

$$18Q = y_0 + y_1 \cos 170° + y_2 \cos 340° + \cdots y_{35} \cos 5950°$$

$$18a = y_1 \sin 10° + y_2 \sin 20° + \cdots y_{35} \sin 350°$$

$$18b = y_1 \sin 20° + y_2 \sin 40° + \cdots y_{35} \sin 700°$$
$$\vdots$$
$$18q = y_1 \sin 170° + y_2 \sin 340° + \cdots y_{35} \sin 5950°$$

The sines and cosines of the multiple angles in the above formulæ may all be expressed in terms of the cosines of angles in the first quadrant by means of the well-known trigonometrical relations.

When similar terms are combined A, B, $C \cdots Q$, the coefficients of the cosine terms, and a, b, $c \cdots q$, the coefficients of the sine terms in the Fourier series will appear as the sum of a series of products of partial coefficients (A_0, A_1, $A_2 \cdots A_8$; B_0, $B_2 \cdots B_8$; C_0, C_3, C_6; etc., and similarly for small letters) and cosines of the multiples of 10° in the first quadrant.

To avoid suffices so far as possible denote y_0, y_1, $y_2 \cdots y_{35}$ by (0), (1), (2) $\cdots$ (35) respectively.

A, B, $C \cdots Q$, then denote the coefficients respectively in the 1st, 2d, 3d $\cdots$ 17th frequency terms of the cosines and a, b, $c \cdots q$ those of the sines. The capitals and small letters affected with suffices denote the corresponding partial coefficients, the suffix itself denoting the multiple of 10° occurring in the term.

The expressions for the coefficients may then be presented as follows:

	18A			18a
A_0	(0) —(18)	cos 0°	a_0	(9) —(27)
A_1	(1)— 17)—(19)+(35)	cos 10°	a_1	(8)+(10)—(26)—(28)
A_2	(2)—(16)—(20)+(34)	cos 20°	a_2	(7)+(11)—(25)—(29)
A_3	(3)—(15)—(21)+(33)	cos 30°	a_3	(6)+(12)—(24)—(30)
A_4	(4)—(14)—(22)+(32)	cos 40°	a_4	(5)+(13)—(23)—(31)
A_5	(5)—(13)—(23)+(31)	cos 50°	a_5	(4)+(14)—(22)—(32)
A_6	(6)—(12)—(24)+(30)	cos 60°	a_6	(3)+(15)—(21)—(33)
A_7	(7)—(11)—(25)+(29)	cos 70°	a_7	(2)+(16)—(20)—(34)
A_8	(8)—(10)—(26)+(28)	cos 80°	a_8	(1)+(17)—(19)—(35)

where it is to be understood:

$$18A = A_0 \cos 0° + A_1 \cos 10° + A_2 \cos 20° + \cdots A_8 \cos 80°$$

$$18a = a_0 \cos 0° + a_1 \cos 10° + a_2 \cos 20° + \cdots a_8 \cos 80°$$

also that, *e. g.*,

$$A_4 = (4) - (14) - (22) + (32) = y_4 - y_{14} - y_{22} + y_{32}$$

and

$$a_6 = (3) + (15) - (21) - (23) = y_3 + y_{15} - y_{21} - y_{23}$$

18*B*

$$B_0 = \begin{cases} (0)+(18) = B_0' \\ -(9)-(27) = -B_0'' \end{cases}$$

$$B_2 = \begin{cases} (1)+(35)+(17)+(19) = B_2' \\ -(8)-(28)-(10)-(26) = -B_2'' \end{cases}$$

$$B_4 = \begin{cases} (2)+(34)+(16)+(20) = B_4' \\ -(7)-(29)-(11)-(25) = -B_4'' \end{cases}$$

$$B_6 = \begin{cases} (3)+(33)+(15)+(21) = B_6' \\ -(6)-(30)-(12)-(24) = -B_6'' \end{cases}$$

$$B_8 = \begin{cases} (4)+(32)+(14)+(22) = B_8' \\ -(5)-(31)-(13)-(23) = -B_8'' \end{cases}$$

18*b*

$$b_1 = \begin{cases} (4)-(32)-(14)+(22) = b_1' \\ +(5)-(31)-(13)+(23) = b_1'' \end{cases}$$

$$b_3 = \begin{cases} (3)-(33)-(15)+(21) = b_3' \\ +(6)-(30)-(12)+(24) = b_3'' \end{cases}$$

$$b_5 = \begin{cases} (2)-(34)-(16)+(20) = b_5' \\ +(7)-(29)-(11)+(25) = b_5'' \end{cases}$$

$$b_7 = \begin{cases} (1)-(35)-(17)+(19) = b_7' \\ +(8)-(28)-(10)+(26) = b_7'' \end{cases}$$

where it is to be understood, *e. g.*, that

$$B_0 = B_0' - B_0'', \text{ etc.},$$

and

$$b_3 = b_3' + b_3'', \quad b_5 = b_5' + b_5'', \quad \text{etc.}$$

18*D*

$$D_0 = B_0' + B_0''$$

$$D_2 = -B_8' - B_8''$$

$$D_4 = B_2' + B_2''$$

$$D_6 = -B_6' - B_6''$$

$$D_8 = B_4' + B_4''$$

18*d*

$$d_1 = b_5' - b_5''$$

$$d_3 = b_3' - b_3''$$

$$d_5 = b_7' - b_7''$$

$$d_7 = b_1' - b_1''.$$

The partial coefficients of the other components may all now be expressed in terms of those just obtained for A, B, D, a, b, d.

For convenience they are arranged in tabular form and the partial coefficients A_0, A_1, M_0, M_1, etc., are indicated by suffices only under the letter to which they belong.

A	Q	E	M	G	K	a	q	e	m	g	k
0	0	0	0	0	0	0	0	0	0	—0	—0
1	—1	5	—5	7	—7	1	—1	5	—5	—7	7
2	2	—8	—8	—4	—4	2	2	—8	—8	4	4
3	—3	—3	3	—3	3	3	—3	—3	3	3	—3
4	4	—2	—2	8	8	4	4	—2	—2	—8	—8
5	—5	—7	7	1	—1	5	—5	—7	7	—1	1
6	6	6	6	6	6	6	6	6	6	—6	—6
7	—7	1	—1	—5	5	7	—7	1	—1	5	—5
8	8	4	4	—2	—2	8	8	4	4	2	2

Thus the partial coefficients of Q, E, M, G, and K are the same as those of A but with different signs and in a different order, *e. g.*:

$$A_2 = Q_2 = -E_8 = -M_8 = -G_4 = -K_4,$$
$$a_5 = -q_5 = -e_7 = m_7 = -g_1 = k_1.$$

In like manner those of B, J and N are the same; also P, H and D.

B	J	N	P	H	D	b	j	n	p	h	d
0	0	0	0	0	0						
2	—8	—4	2	—8	—4	1	5	—7	—1	—5	7
4	—2	8	4	—2	8	3	—3	3	3	—3	3
6	6	6	6	6	6	5	—7	—1	—5	7	1
8	4	—2	8	4	—2	7	1	5	7	1	5

$$C_0 = A_0 - A_6 = O_0, \qquad c_0 = -a_0 + a_6 = o_0,$$
$$C_3 = A_1 - A_5 - A_7 = -O_3, \qquad c_3 = -a_1 + a_5 + a_7 = -o_3,$$
$$C_6 = A_2 - A_4 - A_8 = O_6, \qquad c_6 = -a_2 + a_4 + a_8 = o_6,$$

$$F_0 = B_0 - B_6 \qquad L_0 = D_0 - D_6, \qquad I_0 = C_0 - C_6,$$
$$F_6 = B_2 - B_4 - B_8, \qquad L_6 = -D_2 + D_4 + D_8, \qquad i_0 = -c_0 + c_6.$$
$$f_3 = -b_1 + b_5 + b_7, \qquad l_3 = -d_1 + d_5 + d_7,$$

The results thus obtained for the partial coefficients may now be easily embodied in a simple scheme, which may be checked at every point by the values given above. The working of this scheme is exhibited in the case of a particular example (given in full, with values of measured ordinates, on pages 268, 269) as follows:

Scheme.

Computation of Partial Coefficients (cosine terms.) (Left-hand side of computation sheet.)

		(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(1)[1]	y's	35	62	57	44	28	17	15	20	25	28
(2)[1]	y's		17	8	9	23	39	44	44	43	43
(3)		35	79	65	53	51	56	59	64	68	71
(4)[2]		29	64	70	75	81	80	74	71	72	−15[3]
(5)	A's	6	15	−5	−22	−30	−24	−15	−7	−4	−15[3]

		(0)	(2)	(4)	(6)	(8)
(6)	B'	64	143	135	128	132
(7)	B''	71	140	135	133	136
(8)	B's	−7	3	0	−5	−4
(9)		5	0			
(10)			4			

								(0)	(3)	(6)
(11)	F's	−2	7				C's	21	46	29
		(0)	(4)	(8)	−(6)	−(2)	$-C_6$	−29		
(12)	D's	135	283	270	261	268	I_0	− 8(÷18 = coeff.)		
(13)		261	270							
(14)			268							
(15)	L's	396	821	$L_0+L_6=1217=36(\bar{y})$.						

[1] Line (1) contains the ordinates $y_0, y_1, \cdots y_9$. Line (2) contains the ordinates $y_{35}=17$, $y_{34}=8$, $y_{33}=9$, etc., etc.

[2] Line (4) contains sums y_{18}, $y_{17}+y_{19}$, $y_{16}+y_{20}$, etc., from right-hand side of computation sheet.

[3] Difference belonging to the right-hand side of computation sheet.

Scheme (*continued*).

Computation of partial coefficients (sine terms). (Right-hand side of computation sheet.)

(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	
28[1]	27	23	25	37	48	56	53	44		(1)
43[1]	45	48	49	43	33	19	17	20	29	(2)
71[1]	−18	−25	−24	− 6	15	37	36	24		(3)
−15	−18	−24	−29	−22	5	35	49	45		(4)
−15	−36	−49	−53	−28	20	72	85	69	a's	(5)

					(1)	(3)	(5)	(7)		
					−10	−2	13	21	b'	(6)
(0)	(3)	(6)			−16	−5	1	0	b''	(7)
15	36	49			−26	−7	14	21	b's	(8)
72	20	−28						14	b_5	(9)
	85	69						26	$-b_1$	(10)
87	141	90	c's					61	f_3	(11)
		−87	$-c_0$		(7)	(3)	(1)	(5)		
(÷18=coeff.)		3	i_0		6	3	12	21	d's	(12)
								−12	$-d_1$	(13)
								6	$+d_7$	(14)
								15	l_3	(15)

[1] Ordinates and sum from the left-hand side of computation sheet.

EXPLANATION OF COMPUTATION OF PARTIAL COEFFICIENTS.

Cosine Terms.

Arrange the measured ordinates $y_0, y_1, y_2 \cdots y_{35}$ in two horizontal lines (1) and (2) across the computation sheet putting $y_0 \cdots y_{17}$ in line (1), and $y_{18} \cdots y_{35}$ in reversed order in line (2) so that y_{35} is under y_1, y_{34} under y_2, and so on.

Left-hand Side:

1. Add line (1) to line (2) putting sums under columns (0) $\cdots$ (9) in line (3) and sums (10) $\cdots$ (18) reversed in line (4) so that sum in column (10) is under sum of column (8), (11) under (7), etc., *i. e.*, on left-hand side of the sheet.

2. Subtract line (4) from (3), the differences in order in (5) are the partial coefficients, $A_0, A_1, \cdots A_8$, which are subsequently to be multiplied into cos 0°, cos 10° $\cdots$ cos 80° respectively and the products added to get $18A$.

3. Add (3) to (4) putting sums (0) $\cdots$ (4) in line (6), and (9) $\cdots$ (5) respectively under them in line (7). Subtract line (7) from (6), the differences in order in (8) are B_0, B_2, B_4, B_6, B_8, subsequently to be multiplied into cos 0°, cos 20°, $\cdots$ cos 80° respectively.

4. Arrange the B's in two columns as indicated in lines (9) and (10), reversing the signs of B_4, B_6, B_8. The algebraic sums will be F_0 and F_6 as shown in line (11).

5. Add (6) to (7), the sums in line (12) are in order, $D_0, D_4, D_8, -D_6$ and $-D_2$.

6. Arrange the D's in two columns as indicated in lines (13) and (14). The sums in (15) are the L's.

7. Arrange the A's in three columns as indicated in lines (8), (9) and (10), reversing the signs of $A_4, A_5 A_6, A_7, A_8$. The algebraic sums will be the C's in line (11).

8. Form $I_0 = C_0 - C_6$ under C_0 in line (12).

Sine Terms.

Right-hand Side:[1]

1. Subtract line (2) from (1) placing differences in (3) in columns marked (10) $\cdots$ (17) and differences (9) $\cdots$ (1) under them in line (4) so that (9) is by itself and (8) is under (10), (7) under (11), etc.

2. Add line (3) to (4) putting sums in columns (9) $\cdots$ (17). The sums in order are the partial coefficients $a_0, a_1 \cdots a_8$, which are

[1] It is interesting and instructive to compare these directions with those given above for the "Left-hand Side."

subsequently to be multiplied into cos 0°, cos 10° ··· cos 80° respectively and the products added to get 18*a*.

3. Subtract (3) from (4) placing differences (14) ··· (17) in line (6), and differences (13) ··· (10) under them respectively in line (7). Add line (6) to (7), the differences in order in (8) are b_1, b_3, b_5, b_7, subsequently to be multiplied into cos 10°, cos 30°, cos 50°, cos 70° respectively.

4. Form $f_3 = b_7 + b_5 - b_1$ to the right in lines (8), (9), (10) and (11).

5. Subtract line (7) from (6), the differences in line (12) are in order, d_7, d_3, d_1 and d_5.

6. Form $l_3 = d_5 - d_1 + d_7$ in lines (12), (13), (14) and (15).

7. Form $c_0 = -a_0 + a_6 = o_0$, $c_3 = -a_1 + a_5 + a_7 = -o_3$ and $c_6 = -a_2 + a_4 + a_8 = o_6$ to the left in lines (8) ··· (11).

8. Form $i_0 = -c_0 + c_6$ under c_6 in line (12).

The partial coefficients for E, G, K, M and Q are the same as those obtained in line (5) for A, but with different signs and in a different order. A and Q are in the same order with the signs alternately the same and opposite; similarly for the pairs, E and M, G and K.

In a similar way the partial coefficients of B, J and N, and of P, H and D are the same differing only in sign and order; the partial coefficients of C and O differ only in sign. The foregoing remarks apply also to the partial coefficients of the sine terms, substituting small letters for the capitals.

These relations are given in compact form for reference in the tables on page 262 and suggest the following scheme for performing the necessary multiplications by the cosines of the multiples of 10° and the additions to obtain the final coefficients:

SCHEME (*continued*).

Multiplication by the Cosines.

		(1)		(2)		(3)		*Natural Cosines.*		
	A	*A*	*Q*	*E*	*M*	*G*	*K*	(1)	(2)	(3)
(0)	6	6.0		6.0		6.0		1.000	+1.000	+1.000
(1)	15		14.8		9.6		5.2	.985	+ .643	+ .342
(2)	− 5	−4.7		0.9		3.9		.940	− .174	− .766
(3)	−22		−19.1		19.1		19.1	.866	− .866	− .866
(4)	−30	−23.0		28.2		−5.2		.766	− .940	+ .174
(5)	−24		−15.4		8.2		−23.7	.643	− .342	+ .985
(6)	−15	−7.5		−7.5		−7.5		.500	+ .500	+ .500
(7)	− 7		−2.4		−6.9		4.5	.342	+ .985	− .643
(8)	− 4	−0.7		−3.1		3.8		.174	+ .766	− .940
Sums		−29.9	−22.1	+24.5	+30.0	+1.0	+5.1			
Coeff.		−52.0	− 7.8	+54.5	−5.5	+6.1	−4.1	(to be divided by 18.)		

	B	(1) B	(2) J	(3) N	Natural Cosines. (1)	(2)	(3)	(1) P	(2) H	(3) D	D	
(0)	−7	−7.0	−7.0	−7.0	1.000	+1.000	+1.000	135	135.0	+135.0	+135	(0)
(2)	+3	2.8	−0.5	−2.3	.940	−.174	− .766	−266	49.2	+216.8	−283	−(4)
(4)	0	0.0	0.0	0.0	.766	−.940	+ .174	+206.8	−253.8	+46.8	+270	(8)
(6)	−5	−2.5	−2.5	−2.5	.500	+.500	+ .500	−130.5	−130.5	−130.5	−261	(6)
(8)	−4	−0.7	−3.1	+3.7	.174	+.766	− .940	+46.6	+205.3	−251.9	+268	−(2)
Coeff.		−7.4	−13.1	−8.1	(÷18)			−8.1	+5.2	+16.2	(÷18)	

	C	C	O	cos.	c	o	c	F	cos.	L	f	l
(0)	21	21.0	21.0	1.000	87.0	87.0	87	−2.0	1.000	396.0		
(3)	46	39.8	−39.8	+.866	−122.1	122.1	141	3.5	.500	−410.5	61	.866 15
(6)	29	14.5	14.5	.500	45.0	45.0	90					
Coeff.		75.3	−4.3		254.1	9.9		+1.5		−14.5	52.8	13.0

MULTIPLICATION BY THE COSINES.

Left-hand Side and Middle:

Since the coefficients are the same, in the same order and with alternately the same and opposite signs in the pairs A, Q: E, M; Q, R, but the order different in the several pairs, the natural cosines are arranged in three different orders in columns (1), (2) and (3) in such a way and with such signs that, being multiplied by the partial coefficients A in order, the proper products are obtained for Q, E, M, G and K.

Similar remarks, *mutatis mutandis*, apply to the calculation of the remaining coefficients for both cosine and sine terms. The processes will be readily understood by comparing carefully the tables on page 262 with the following scheme:[1]

1. Place the partial coefficients $A_0 A_1 \cdots A_8$ in the left-hand column. Multiply them by the corresponding cosines in the same line in columns (1), (2) and (3), putting the products (1) alternately in A and Q columns; (2) in a similar way in E and M columns; (3) in G and K columns.

2. Since each pair has the same coefficients in the same order but alternating in sign, the sum of the A and Q columns $-29.9-22.1=-52.0=18A$, their difference, $-29.9+22.1=-7.8=18Q$, similarly for the other pairs.

3. Place the partial coefficients B_0, B_2, B_4, B_6, B_8, in the left-hand column. Multiply them by the corresponding cosines in the same line in columns (1), (2) and (3), placing the products in columns B,

[1] Explanations are given here only for A, B, C, etc. The processes on the right hand side for a, b, c, etc., are entirely similar.

J, N respectively. The sums will give $18B$, $18J$ and $18N$, respectively equal to -7.4, -13.1, -8.1.

4. For convenience the D's must be arranged in the right-hand column in the order D_0, $-D_4$, D_8, D_6 and $-D_2$. Proceeding as in (3) $18P=-8.1$, $18H=+5.2$ and $18D=+16.2$ are obtained.

5. C, O, F and L are similarly obtained as shown.

6. Dividing the results thus obtained in the lines marked "Coeff." by 18, the coefficients A, B, C, etc., and a, b, c, etc., are obtained and may be substituted in the following (approximate) form of Fourier's Theorem:

$$y=(\bar{y})^1+A\cos x+B\cos 2x+C\cos 3x+\cdots Q\cos 17x^2 \\ +a\sin x+b\sin 2x+c\sin 3x+\cdots q\sin 17x^2$$

The phase of each simple harmonic may then be computed from the formulæ: $\tan\varphi_1=A/a$, $\tan\varphi_2=B/b$, etc., and the amplitudes from the formulæ,

$$a_1=\sqrt{a^2+A^2},\quad a_2=\sqrt{b^2+B^2},\quad \text{etc.},$$

in the usual way.[3]

CASE, $2n=12$. GIVING FREQUENCIES 1–5.

When only a few simple harmonics are present the case in which $2n=12$, requiring 12 given ordinates, may be applied.

As this case is of frequent occurrence it seems desirable to append the values of the coefficients and the scheme of computation. The notation is the same as that already used.

6A			6a	
$A_0=(0)-(6)$	$=E_0$	$\cos\ 0°$	$a_0=(3)-(9)$	$=e_0$
$A_1=(1)-(5)-(7)+(11)$	$=-E_1$	$\cos 30°$		
			$a_1=(2)+(4)-(8)-(10)$	$=-e_1$

[1] The term $(\bar{y})$ depends on the choice of the axis of x and may generally be omitted. Its value is $(\bar{y})=\overline{L_0+L_6}/36=\Sigma y/36$, *i. e.*, the mean value of the 36 y's.

[2] 36 divisions will give the first 17 frequencies or simple periods. $2n$ divisions will give the first $n-1$ coefficients. The formulæ give illusory results for the remaining $n+1$ coefficients.

[3] Dr. Bevier prepared a printed blank for use in his investigations in the "Acoustic Analyses of the Vowels" in accordance with the scheme presented here. In it he printed all the signs of the products to be inserted as—. This lessens the labor, and promotes accuracy by calling special attention to every sign; the symbol — being readily changed to + when necessary. A copy of this blank with a special example worked out is given on pages 268, 269.

x	y
0	35
1	62
2	57
3	44
4	28
5	17
6	15
7	20
8	25
9	28
10	27
11	23
12	25
13	37
14	48
15	56
16	53
17	44
18	29
19	20
20	17
21	19
22	33
23	43
24	49
25	48
26	45
27	43
28	43
29	44
30	44
31	39
32	23
33	9
34	8
35	17

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	
	35	62	57	44	28	17	15	20	25	28	27	23	25	37	48	56	53	44		
		17	8	9	23	39	44	44	43	43	45	48	49	43	33	19	17	20	29	
	35	79	65	53	51	56	59	64	68	71	—18	—25	—24	— 6	15	37	36	24		
	29	64	70	75	81	80	74	71	72	—15	—18	—24	—29	—22	5	35	49	45		
A's	6	15	— 5	—22	—30	—24	—15	— 7	— 4	—15	—36	—49	—53	—28	20	72	85	69		a's
	64	143	135	128	132										—10	—2	13	21		
	71	140	135	133	136										—16	—5	1	0		
B's	—7	3	0	—5	—4		6	15	—5	—15	36	49			—26	—7	14	21		b's
	5	0					15	24	30	72	20	—28						14		
		4						7	4		85	69						26		
F's	—2	7				C's	21	46	29	87	141	90 c's						61		f's
							—29					—87								
D's	135	283	270	261	268										6	3	12	21		d's
	261	270					— 8	(l_0)			(i_0)	3						—12		
		268																6		
L's	396	821																15		l's

A	**1**	**17**	**5**	**13**	**7**	**11**	Natural Cosines	**1**	**17**	**5**	**13**	**7**	**11**	*a*
+ 6	+ 6.0		+ 6.0		+6.0		**1000** + **1000** + **1000**		—15.0		—15.0		—15.0	—15
+15		+14.8		+ 9.6		+ 5.2	**.985** + **.643** + **.342**	—35.5		— 23.1		—12.3		—36
— 5	— 4.7		+ 0.9		+3.9		**.940** — **.174** — **.766**		—46.1		+ 8.5		+37.5	—49
—22		—19.1		+19.1		+19.1	**.866** — **.866** — **.866**	—45.9		+ 45.9		+45.9		—53
—30	—23.0		+28.2		—5.2		**.766** — **.940** + **.174**		—21.4		+ 26.3		— 4.9	—28
—24		—15.4		+ 8.2		—23.7	**.643** — **.342** + **.985**	+12.9		— 6.8		+19.7		+20
—15	— 7.5		— 7.5		—7.5		**.500** + **.500** + **.500**		+36.0		+ 36.0		+36.0	+72
— 7		— 2.4		— 6.9		+ 4.5	**.342** + **.985** — **.643**	+29.1		+ 83.7		—54.7		+85
— 4	— 0.7		— 3.1		+3.8		**.174** + **.766** — **.940**		+12.0		+ 52.9		—64.9	+69
	—29.9	—22.1	+24.5	30.0	1.0	5.1		—39.4	—34.5	+ 99.7	+108.7	— 1.4	—11.3	
Coeff.	— 52.0	— 7.8	+54.5	— 5.5	6.1	— 4.1		—73.9	— 4.9	+208.4	— 9.0	—12.7	+ 9.9	

B	**2**	**10**	**14**	Natural Cosines	**16**	**8**	**4**	*D*	*b*	**2**	**10**	**14**	Natural Cosines	**16**	**8**	**4**	*d*
—7	—7.0	— 7.0	—7.0	**1000** + **1000** + **1000**	+135.0	+135.0	+135.0	+135									
+ 3	+ 2.8	— 0.0	—2.3	**.940** — **.174** — **.766**	—266.0	+ 49.2	+216.8	—283	— 26	—25.6	—16.8	+ 8.9	**.985** + **.643** — **.342**	—5.8	— 3.8	+ 2.1	— 6
—0	—0.0	— 0.0	—0.0	**.766** — **.940** + **.174**	+206.8	—253.8	+ 46.8	+270	— 7	— 6.1	+ 6.1	— 6.1	**.866** — **.866** + **.866**	+2.6	— 2.6	+ 2.6	+ 3
—5	—2.5	— 2.5	—2.5	**.500** + **.500** + **.500**	—130.5	—130.5	—130.5	—261	+14	+ 9.0	— 4.8	—13.8	**.643** — **.342** — **.985**	—7.7	+ 4.1	+11.8	—12
—4	—0.7	— 3.1	+ 3.7	**.174** + **.766** — **.940**	+ 46.6	+205.3	—251.9	+268	+21	+ 7.2	+20.7	+13.5	**.342** + **.985** + **.643**	+7.0	+20.7	+13.4	+21
	—7.4	—13.1	—8.1		— 8.1	+ 5.2	+ 16.2			—15.5	+ 5.2	+ 2.5		—3.9	+18.4	+29.9	

C	**3**	**15**		**3**	**15**	*c*	**12**		**6**	**12**		**6**	**9**
+21	+21.0	+21.0	**1000**	+ 87.0	+ 87.0	+ 87							I_0, —8
+46	+39.8	—39.8	+ **.866** —	+122.1	—122.1	+141	+396.0	**1000**	—2.0	+15.0	**.866**	+61	
+29	+14.5	+14.5	**.500**	+ 45.0	+ 45.0	+ 90	—410.5	**.500**	+3.5				i_0, +3
	+75.3	— 4.3		254.1	+ 9.9		— 14.5		+1.5	+12.9		+52.8	

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**
R	—2.8	—0.4	+ 4.2	+0.9	+ 3.0	+0.1	+0.3	+0.3	—0.4	—0.7	—0.2	—0.8	—0.3	—0.5	—0.2	—0.5	—0.4
r	—4.1	—0.9	+14.2	+1.7	+11.6	+2.9	—0.7	—1.0	+0.2	+0.3	+0.5	+0.7	—0.5	+0.1	+0.6	—0.2	—0.3

The numbers 1–17 in black faced type above indicate the orders of the simple harmonics present in the compound harmonic curve.

$$A_2=(2)-(4)-(8)+(10) \quad =E_2\,|\cos 60°|\quad a_2=(1)+(5)-(7)-(11)=e_2$$

6B

$$B_0=\begin{cases}(0)+(6)= & B_0'\\ -(3)-(9)=- & B_0''\end{cases}$$

6b

$$b_1=\begin{cases}(1)-(5)+(7)-(11)=b_1'\\ +(2)-(4)+(8)-(10)=b_1''\end{cases}$$

$$B_2=\begin{cases}(1)+(5)+(7)+(11)= & B_2'\\ -(2)-(4)-(8)-(10)=- & B_2''\end{cases}$$

6C

$$C_0=A_0-A_2$$

6c

$$c_0=-a_0+a_2$$

6D

$$D_0=B_0'+B_0''$$

$$D_2=-B_2'-B_2''$$

6d

$$d_1=b_1'-b_1''$$

where $B_0'=y_0+y_6$, $B_0''=y_3+y_9$, etc.,

and $b_1'=y_1-y_5+y_7-y_{11}$, $b_1''=y_2-y_4+y_8-y_{10}$, etc.

As a particular example we take every third ordinate of the series of 36 ordinates given in the case $2n=36$.

(0)	(1)	(2)	(3)	(4)	(5)	(6)	
35	44	15	28	25	56		
	9	44	43	49	19	29	
35	53	59	71	−24	37		
29	75	74	−15	−29	35		
A's 6	−22	−15	−15	−53	72		a's
64	128			5	2		
71	133	C_0	c_0		5		
B's −7	−5	6	72		−7		b's
		15	−15				
D's 135	−261	(Coeff.) 21	87	(Coeff.)	3		d's

A	1	5	cos.	1	5	a	b	2	cos.
6	6.0		1.000		−15.0	−15	−7	−6.1	.866
−22		−19.1	.866	−45.9		−53	d	4	
−15	−7.5		500		36.0	72	3	2.6	.866
	−1.5	−19.1		−45.9	21.0				
(Coeff.)[1]	−20.6	+17.6		−24.9	66.9				

B	2	cos.	4	D		
−7	−7.0	1.000	135.0	135		
−5	−2.5	.500	−130.5	−261	3	3
(Coeff.)[1]	−9.5		+4.5		21	87

	1	2	3	4	5
$R=$	−3.4	−1.6	+ 3.5	+0.8	+ 2.9
$r=$	−4.2	−1.0	+14.5	+0.4	+11.2

RUTGERS COLLEGE, NEW BRUNSWICK, N. J.

[1] The results in line marked (Coeff.) must be divided by 6.

THE VOWEL I (AS IN PIQUE)

By Louis Bevier, Jr.

IN studying the vowel *i* (as in *pique*) we do not meet the difficulties noted in dealing with *i*ᵛ. The former is a sound which occurs in our speech in both closed syllables and open syllables, both long and short, and any singing voice can intone it truly at ordinary pitches without difficulty. This vowel is more closely articulated than any of those considered in the foregoing articles, and stand as the terminus of close palatal articulation. In general both physiologically and acoustically the sounds *a*, *a*ᵉ, *ę*, *i*ᵉ, *i*, with their numberless possible intermediates, form a true series.

On the other hand a difficulty of another sort arises in studying *i* from the phonographic record. The amplitude of the high partial which chiefly characterizes it is not very large, and on low fundamentals, even with the great enlargement produced by my apparatus, it is often too small for accurate measurement. Hence, I give in the table no chord-tone with a frequency less than 160 to the second. On higher fundamentals this difficulty disappears.

In the table (p. 272) will be found a sufficient number of analyses to afford the basis for an acoustic definition.

The phenomena presented in the table are for the most part not difficult to interpret. It is quite obvious at a glance that the vowel *i* is characterized by a high-pitched upper partial, powerfully reinforced, with a frequency of about 2,000 or 2,100, *i. e.*, the maximum point of this uppermost region of resonance generally centers at about 2,050. This does not however exclude considerable freedom in different voices, and even in different utterances of the same voice. When the vowel is more closely articulated the pitch rises, when more openly articulated it falls. Consequently a range from 1,900 to 2,500 may easily be established when a large number of examples are analyzed, *i. e.*, a range a little greater than a major third, much

ANALYSIS OF THE *I*-CURVE.

		I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII
		160	320	480	640	800	960	1120	1280	1440	1600	1760	1920	2080	2240	2400	2560	2720
6	126.2	29.2	11.4	3.5	5.6	1.0	2.2	0.4	1.2	4.0	3.9	5.8	**16.4**	6.7	4.9	2.1	1.0	0.7
		192	364	576	768	960	1152	1344	1536	1728	1920	2112	2304	2496	2688	2880	3072	3264
6	112.4	26.4	15.7	2.8	1.9	1.4	0.3	1.3	3.7	9.1	**16.5**	8.2	3.1	3.1	2.4	2.0	1.0	1.1
		202	404	606	808	1010	1212	1414	1616	1818	2020	2222	2424	2626	2828	3030	3232	3434
1	95.0	58.4	6.8	4.9	2.1	3.1	2.5	1.9	1.2	1.9	**6.6**	1.2	0.8	0.6	1.9	1.4	0.8	1.9
		226	452	678	904	1130	1356	1562	1808	2034	2260	2486	2712	2938	3164	3390	3616	3842
2	123.6	55.8	3.6	6.5	5.3	0.5	7.1	3.4	5.8	**15.5**	0.8	4.7	1.8	2.3	0.3	1.5	0.6	0.5
1	142.2	63.8	8.6	0.3	1.5	0.6	1.4	0.6	2.7	**11.8**	1.8	1.4	1.7	1.3	1.4	0.6	0.6	0.9
		256	512	768	1024	1280	1536	1792	2048	2304	2560	2816	3072	3328	3584	3840	4096	4352
1	110.0	33.9	2.4	0.6	2.9	2.7	5.0	2.9	6.3	**19.7**	2.0	5.5	4.9	6.3	3.9	0.6	0.3	0.5
1	83.0	43.4	2.4	0.9	6.0	3.1	2.2	5.1	**14.7**	6.3	1.0	6.5	0.7	2.9	3.1	0.7	0.5	0.3
1	197.8	51.7	1.3	2.1	5.1	4.0	4.4	5.0	**13.3**	0.2	3.8	0.5	2.4	3.3	0.1	1.1	1.1	0.6
6	94.7	35.3	5.2	0.4	2.3	3.3	6.5	6.1	**14.5**	**12.9**	4.8	3.1	2.2	0.9	0.8	0.8	0.1	0.8
5	38.7	42.6	6.4	7.5	1.0	6.7	1.0	2.6	2.3	**3.9**	1.9	3.4	**7.0**	4.9	1.3	3.9	3.1	0.5
		272	544	816	1088	1360	1632	1904	2176	2448	2720	2992	3264	3538	3808	4080	4352	4624
1	37.9	39.6	5.6	5.6	5.3	4.0	4.0	8.0	**11.9**	2.6	2.1	3.7	1.3	1.1	1.6	1.6	1.0	1.0
5	33.2	33.2	6.3	3.0	6.2	4.0	3.4	**21.2**	3.0	3.7	6.0	1.9	1.9	1.0	0.9	0.8	0.1	0.4
		320	640	960	1280	1600	1920	2240	2460	2880	3200	3520	3840	4160	4480	4800	5120	5440
5	40.0	32.0	6.5	2.0	5.0	3.0	3.0	3.5	3.0	9.0	**12.5**	9.0	2.5	2.0	2.0	2.0	2.0	1.0
7	64.4	27.9	7.9	2.2	4.0	6.1	9.2	**26.5**	5.3	1.1	4.3	0.9	0.8	0.5	0.8	1.7	0.5	0.3
		384	768	1152	1536	1920	2308	2692	3072	3456	3840	4224	4608	4992	5376	5760	6144	6528
5	62.7	11.6	6.7	6.2	6.9	5.9	8.6	**20.0**	7.8	11.8	2.8	2.6	2.8	1.6	2.1	0.7	0.8	1.1
7	130.1	13.1	5.8	5.4	10.8	11.1	**30.7**	6.9	5.3	3.2	2.2	1.1	1.5	0.1	1.1	0.2	0.9	0.6
8	30.2	31.8	3.6	6.6	6.3	**20.2**	**13.9**	2.7	4.3	2.6	0.7	0.7	1.3	1.3	1.3	1.0	0.7	1.0
10	20.9	28.7	8.1	2.4	7.1	6.6	10.1	**22.0**	2.0	4.7	1.4	1.4	1.0	1.0	1.0	1.0	0.5	1.0
		512	1024	1536	2048	2560	3072	3584	4096									
5	74.8	19.3	10.4	15.8	10.8	7.6	**21.7**	7.0	4.4									
7	167.7	22.2	12.1	21.8	**29.0**	7.8	3.3	1.7	2.1									
8	70.3	45.9	13.1	10.2	**14.1**	10.0	2.7	1.6	2.4									
		640	1280	1920	2560	3200	3840	4480	5120									
10	129.6	82.9	1.0	**11.4**	3.0	0.7	0.4	0.3	0.3									

wider than for any of the vowels yet studied. Moreover some exceptional utterances of children's voices (*e. g.*, 5,256 and 5,320) have a strong resonance as high as 3,000 or even 3,200. Such a high-pitched vowel is still heard as *i*, merely sounding somewhat "thin" and "close." If in ordinary speech this close *i* were regularly heard in some words, and a more open one in others, so marked a difference might easily be the basis of a vowel discrimination, but since these are merely exceptional divergences, and since *i* is the "closest" of the palatal series the difference is practically neglected, and passes without notice in the rapid current of speech. The attentive ear however easily notes it, and it requires no long practice in such observation to convince one that some speakers habitually articulate *i* more closely than others.

Examining our table again the next noteworthy fact is the almost complete absence of resonance below this high upper partial. From 1,900 downward there is no considerable resonance until we come either to the chord-tone or to the neighborhood of 400 vibrations to the second. Of course on high chord-tones the absence of resonance throughout this region is not so complete for reasons sufficiently explained in previous articles, but even in these cases the upper partials lying between the points specified are comparatively unimportant. How low the resonance between 500 and 1,900 is on ordinary fundamentals is but realized by computing averages. On chord-tones below and including 320 the average amplitude percentage for all upper partials falling within this region is but 3.5. The only vowel with which a fair comparison can be made is *i*^{r}, which has, as we have seen, little resonance between 600 and 1,800, but even there the average resonance computed as above is 5.1. With *a*, *a*^{r}, and *ę* no comparison of this sort can be made, because their strong resonance regions fall between these very points.

Again we observe the comparatively large amplitude of the chord-tone, much larger even than for *i*^{r}. Comparing averages once more, the amplitude percentage of the fundamental for all pitches of *i* is 37.6, and for *i*^{r} 28.7. How much greater the contrast is when comparing with *a*, *a*^{r}, or *ę* may be seen by the figures given in the article on *i*^{r}.

In accordance with these observations we may therefore define *i* as containing the following elements:

1. A powerfully reinforced upper partial at some pitch, generally lying between 1,900 and 2,500, with considerable freedom as to its exact frequency. It is far oftenest in the neighborhood of 2,050, which may therefore be called its normal pitch. It will be noted that the American *i* is generally more open than the German *i*, and its characteristic upper partial lower pitched. On the other hand, a much greater frequency is occasionally found and does not destroy the *i* quality, provided other characteristics are present.

2. A chord-tone, which is generally present with a large amplitude, much larger relatively than for any of the more open vowels.

3. With these two tones present a true *i* is further characterized by, comparatively speaking, little intermediate resonance. This fact is very important, and seems to give *i* the peculiar thin timbre which it possesses.

The resonance curve, plotted from all the *i* records computed, is added to show the matter in convenient shape to the eye. It is

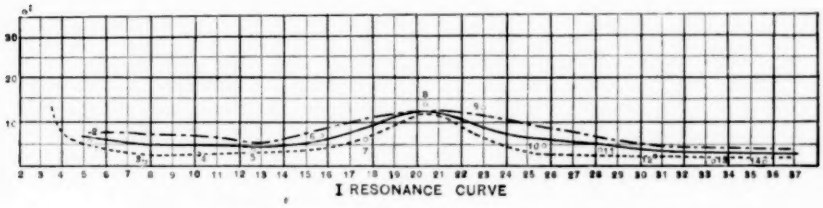

Fig. 1.

clear, however, that the average result gives a too gradual rise and decline for the region of strong resonance, as was the case also for *i*^{e}. It does show truthfully enough the absence of strong resonance between 500 and 1,900, as explained above.

The sheet of curves shows for the most part normal examples, of the usual *i*, *i. e.*, that with strong resonance at or near 2,050, and with a strong fundamental. Voice 6 (256) is a good specimen. The strong fundamental is easily seen by simple inspection, and the eight minor crests reveal upper partial VIII. (*i. e.*, 2,048). The mathematical analysis adds merely a detailed knowledge of the relative amplitudes of the minor partials which are for *i* quite unim-

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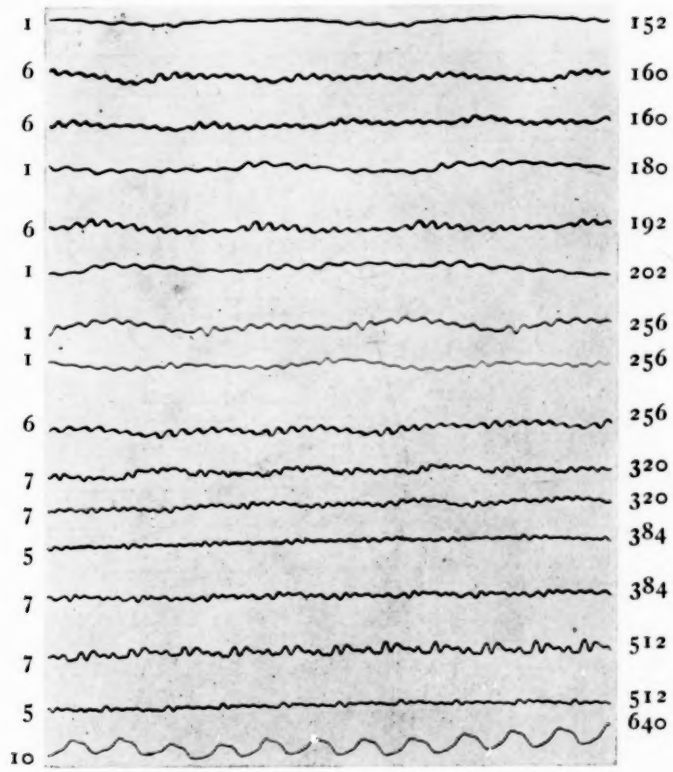

I — Curves

BEVIER.

portant. The examples of voice 5 at 384 and 512 are shown, because they are striking and exceptional in regard to the exceedingly high pitch of the most important upper partial, at 512 for example it is easy to count six minor crests, that is to say VI., with a frequency of 3,072, is the most prominent upper partial.

In concluding this series of studies in the vowels of the palatal series, it may be interesting to exhibit a sheet of curves giving examples of all of these vowels side by side (see Plate). Here are arranged two examples of each vowel. The pitch has been chosen to make the comparsion most instructive, viz., one of each at 256 and one at 512, the octave above. No good example of *a* being at hand at 256, one a semitone higher is shown, which for this purpose answers the same purpose.

It will be noted that of *a* at 272 the fundamental wave has four minor crests superimposed upon it (1,088, cf. PHYS. REV., June, 1900), and at 512 two (1,024); that of a^e at 256 six (1,536, cf. PHYS. REV., March, 1902), and at 512 three (1,536); that of ε at 256 shows interference between VI. and VII. (1,536–1,792, cf. PHYS. REV., April, 1902), and at 512 interference between III. and IV. (1,536–2,048); that of i^e at 256 shows interference between VII. and VIII. (1,792–2,048, cf. PHYS. REV., May-June, 1902), and at 512 IV. predominating over III. (2,048); that of *i* at 256 shows eight minor crests (2,048), and at 512 four (2,048). Details, of course, cannot be gathered from mere inspection. Such a juxtaposition shows, on the one hand, the great similarity of a vowel at one pitch and at the octave above that pitch; and also the continuous rise in pitch of the more characteristic upper partial in passing down from *a* to *i*. It does not show the other less prominent resonances, which are nevertheless, as we have seen, of great importance in the differentiation of the vowels.

It is my expectation to present as soon as practicable similar studies on the labio-guttural vowel-series, from *a* to *u*, including their chief intermediate steps, but these studies will not be ready for publication for some months.

RUTGERS COLLEGE, NEW BRUNSWICK, N. J.

RAYLEIGH'S ALTERNATE CURRENT PHASEMETER.

BY E. S. JOHONNOTT.

LORD RAYLEIGH[1] has described an instrument for measuring the angle of lag of an alternating current behind the E.M.F. impressed at the terminals of the circuit. With proper calibration the readings give also the current, E.M.F., and thus generally such quantities as apparent watts, true watts, power-factor, coefficient of self induction, I^2R loss, hysteresis, etc.

The work described in the following pages was undertaken with the hope of obtaining an instrument suitable for the measurement of the quantities above mentioned by students in the laboratory.

It has been shown by Breslauer[2] that the instrument is particularly well adapted for measurements of activity in circuits of low power-factor such as motors running light and choking-coils with open magnetic circuits. He has also discussed the advantages that it has over the electrodynamometer used as a wattmeter on circuits with low power-factors.

Description and Theory of the Instrument.—The instrument consists of a current coil, M, and an E.M.F. coil, S, arranged as shown in Fig. 1 (p. 281). A piece of soft-iron wire, m, 1 cm. long, is suspended by means of a phosphor-bronze torsion wire, at an angle of 45° to the common axis of the two coils. Each coil is made adjustable along this axis. The deflection of the needle is measured by that of a ray of light reflected from a concave mirror attached to the suspended system.

The couple acting on the needle due to the current in either coil acting alone varies as the square of the current and as sin 2θ, where θ is the angle between the needle and the axis of the coils. The couple thus becomes a maximum for $\theta = 45°$, and zero for $\theta = 0$ and $\theta = 90$.

[1] Phil. Mag., May, 1897, p. 343; Electrician, June, 1897.

[2] Elektrotechnische Zeitschrift, March 13, 1902.

If the currents are of the sine type and $a \cos(\omega t - \phi)$ represents the instantaneous current in M, $b \cos \omega t$ represents the instantaneous current in S, then

$$\left.\begin{aligned} \frac{1}{T}\int_0^T [Ha\cos(\omega t - \varphi)]^2 dt &= \frac{H^2a^2}{2} = A^2 \\ \frac{1}{T}\int_0^T [Kb\cos\omega t]^2 dt &= \frac{K^2b^2}{2} = B^2 \end{aligned}\right\} \quad \text{(i)}$$

where H and K are constants which multiplied into the instantaneous values of the currents give the instantaneous values of the intensities of the magnetic fields produced at the needle by the coils M and S respectively.

Therefore A^2 which is proportional to the mean square value of the current will vary as the reading on the scale, and may be represented by it. Likewise for B^2.

When the currents in the coils act simultaneously giving couples on the needle,

$$\left.\begin{aligned} &(a) \text{ in the same direction, let} \\ &\frac{1}{T}\int_0^T [Ha\cos(\omega t - \varphi) + Kb\cos\omega t]^2 dt = C_1^2 \\ &(b) \text{ in opposite directions, let} \\ &\frac{1}{T}\int_0^T [Ha\cos(\omega t - \varphi) - Kb\cos\omega t]^2 dt = C_2^2 \end{aligned}\right\} \quad \text{(ii)}$$

Then

$$\left.\begin{aligned} C_1^2 &= A^2 + B^2 + 2AB\cos\varphi \\ C_2^2 &= A^2 + B^2 - 2AB\cos\varphi \end{aligned}\right\} \quad \text{(iii)}$$

and C_1^2 and C_2^2 are represented by the readings as before.

Thus, in order to determine the angle of lag, readings are taken with the currents in M and S acting independently and then simultaneously, either in the same or in opposite directions. The computation for $\cos\varphi$ is

$$\cos\varphi = \frac{C_1^2 - A^2 - B^2}{2AB} = \frac{A^2 + B^2 - C_2^2}{2AB} = \frac{C_1^2 - C_2^2}{4AB}. \quad \text{(iv)}$$

Since A^2, B^2, C_1^2 and C_2^2 are simply the readings, no calibrations of the coils are necessary.

From equations (iii) we have

$$C_1^2 + C_2^2 = 2(A^2 + B^2).$$

This relation is very convenient in adjusting the needle to its most sensitive position, viz.: $\theta = 45°$. This is also the position for the least change in the sensibility for a given change in θ. Setting the coils sufficiently near the needle to obtain a large reading for C_1^2, the angular position of the coils relative to the needle is shifted until the equation above is satisfied.

If however θ were greater than 45° where the circuit has a low power-factor, C_1^2 and C_2^2 would each be greater than either A_2 or B_2 and $C_1^2 + C_2^2 > 2(A^2 + B^2)$. The defect found in the instrument by Mr. E. Place[1] may thus very possibly be ascribed to imperfect adjustment.

Methods of Measuring Activity in Circuits having Low Power-factor.—A problem which well illustrates the character of the work which may be undertaken with the phasemeter is the determination of the power-factor and energy losses in choking-coils.

In the results given below two methods have been employed.

Method I., similar to the electrodynamometer used as a wattmeter.

Method II., similar to the electrodynamometer used as a split-phase instrument.

In either case let h and k be constants of the coils, M and S, respectively for any given distances from the needle. Then

$$I = h\sqrt{A^2} = hA$$

$$E = k\sqrt{B^2} = kB.$$

Where I is the current and E is the E.M.F.

Method I.—The E.M.F. coil, S (Fig. 1), is shunted across the circuit and measures the E.M.F. E_0 impressed on the circuit. The coil, M, measures the current. If the angle of lag is determined by eq. (iv) the watts, W_0, consumed in the circuit are given as

$$W_0 = IE_0 \cos \varphi_0.$$

[1] Electrical World, May, 1899, p. 614.

Method II.—An exploring coil exactly similar to the magnetizing coil is wound on the magnetic circuit. The E.M.F. coil is connected in series with the exploring coil and the current coil in series with the magnetizing coil.

If E' be the E.M.F. induced in the exploring coil and φ' the angle of lag between it and the magnetizing current

$$W' = IE' \cos \varphi'$$

will represent the watts spent in the circuit less the I^2R loss in the primary.

In either method it is evident that

$$W = IE \cos \varphi$$

becomes

$$W = kBhA \frac{C_1^2 - C_2^2}{4AB}$$

$$W = \frac{hk}{4} (C_1^2 - C_2^2)$$

$$W = l(C_1^2 - C_2^2)$$

where

$$l = \frac{hk}{4}.$$

In other words the activity, W, in the circuit is proportional to the difference in the readings when the currents act simultaneously, in the first place, in the same direction; in the second place, in opposite directions.

It should be observed that the third expression for the value of $\cos \varphi$ in eq. (iv) is the most accurate for low power-factors.

The wattmeter constant, $l = hk/4$, is at once determined by the current and E.M.F. constants of the instrument. Consequently, even though the current or E.M.F. reading may be small the watts will be represented by the difference of two large readings (see first readings in Table III.). For the same angle of lag on low power-factors the determination will be most accurate when the current and E.M.F. readings are equal.

Since the watts given by Method II. are equal to those consumed in the circuit, less the I^2R loss in the primary, they represent the

hysteresis and eddy current loss provided there is no magnetic leakage and no I^2R loss in the exploring coil.[1] The watts lost in the secondary is never as great as unity in all of the results given below.

In the results given below an attempt has been made to measure the iron losses in a choking-coil whose power-factor was varied through a wide range by opening the magnetic circuit.

An iron anchor-ring was split in halves and wound with magnetizing and exploring coils, each consisting in every case of 200 turns of No. 18 wire.

Two methods of arranging the coils were employed.

1. "Parallel Coils."—Two strands of wire were wound simultaneously on the same form, one constituting the magnetizing and the other the exploring coil. The coils occupied the middle third of one of the halves of the ring.

2. "Distributed Coils."—Twenty small coils each having twenty turns were distributed, ten on each half of the ring. These coils fitted the ring closely and occupied about the middle third of each half. Alternate sections were connected in series and formed the magnetizing coil. The others similarly connected formed the exploring coil. (See Fig. 1.)

Connections.—Fig. 1 shows the connections for using the phasemeter as a wattmeter. T_1 represents the transformer connected to the city mains; T_2, a 1 : 1 transformer consisting of the two sections of the secondary of a T. H. transformer. The auxiliary circuit containing the resistance, Z_1 and the choking-coil, Z_2, was used to alter the conditions in the circuit containing the experimental coil, X. The choking-coil, X, on which the observations were made, was connected in series with the secondary of the 1 : 1 transformer and the current coil of the phasemeter. The dotted lines in the figure show the E.M.F. connections.

In taking the readings by Method I. the terminals of the coil, S, were connected across the magnetizing coil by means of the double-throws witch. Throwing the switch to the terminals (E') of the exploring coil gave the readings for Method II.

[1] See Gray's Ab. Meas. in Elect. and Mag., Vol. II., Chap. XII., p. 683.

Another double-throw switch for including a compensating coil, S', in the E.M.F. circuit while taking the reading, A^2, is shown in the figure. The switch, D, was used to reverse the current through S while taking the readings, C_1^2 and C_2^2.

The coil S consisted of 200 turns of No. 28 manganin wire having a resistance of 660 ohms. A non-inductive resistance of 500 ohms was connected in series with S in all the readings.

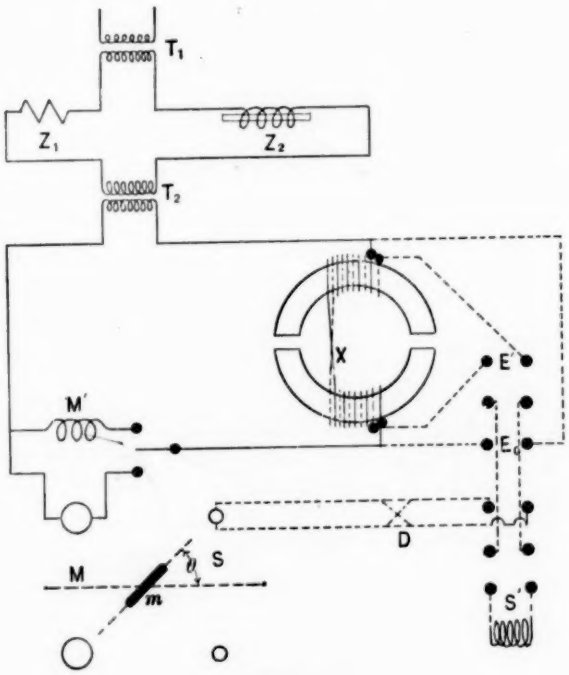

Fig. 1.

The coil M consisted of 3 turns of No. 10 wire and had a compensating coil, M', which was thrown in the line when the reading, B^2, due to the E.M.F. coil was desired.

The Correcting Coil. — In Method I. a correction was made for the component of the current in M which did not flow through the magnetizing coil but passed through S. This was accomplished by winding a fine wire spirally on the wire of M and leading the E.M.F. current through it in a direction opposite to the current in M. This principle is used in such instruments as the Weston wattmeter.

The influence of the correcting coil may be shown as follows: Let φ' be the angle of lag as determined with the phasemeter and φ the true angle. For the term $Ha \cos(\omega t - \varphi)$ in equations (i) and (ii) we must substitute $H[a \cos(\omega t - \varphi) + b \cos \omega t]$ equal say to $Ha' \cos(\omega t - \varphi')$. Then

$$\tan \varphi' = \frac{a \sin \varphi}{a \cos \varphi + b} \qquad \cos \varphi' = \frac{a \cos \varphi + b}{\sqrt{a^2 + b^2 + 2ab \cos \varphi}}.$$

(*a*) When

$$\varphi = 0 \qquad \text{or} \quad \cos \varphi = 1 \qquad \cos \varphi' = 1.$$

In other words no correcting coil is necessary on non-inductive circuits.

(*b*) When $\varphi = \frac{\pi}{2}$ or $\cos \varphi = 0$

$$\cos \varphi' = \frac{b}{\sqrt{a^2 + b^2}} = \frac{1}{\sqrt{\frac{a^2}{b^2} + 1}} = \frac{b}{a}$$

approximately, since in all cases b is small as compared with a. In the results given below b is about $\frac{1}{40}$ ampère while a is $\frac{1}{2}$ or more. Thus at a very low power-factor, without the correcting coil, the watts read by method I. would be too high by about $IE\,b/a$, which might be a large percentage of the actual watts consumed.

Given the currents in the current and E.M.F. coils it is evident that the correcting coil is effective no matter what the positions or relative number of turns of the coils may be.

To insure that the readings were taken under uniform conditions, a second phasemeter consisting of the E.M.F. coil alone was connected across the terminals of the city transformer. This was sensitively adjusted with its spot of light conveniently located on the scale from which the readings were taken.

Results. — In the following tables a constant induced E.M.F. E', was maintained in the exploring coil while the air-gap was opened.

The induction, and consequently the loss of energy by hysteresis, should have remained nearly constant.

In the first column is given the air-gap on each side of the ring in millimeters; in the second, A^2 the reading due to the current coil alone; in the third, the current $I = hA$; in the fourth, the I^2R loss in the magnetizing coil; in the fifth, sixth, seventh and eight, the readings B^2, $C_1^2 - C_2^2$ and the quantities $\cos \varphi'$ and W' for Method II.; in the remaining columns, similar quantities for Method I. together with the quantity $W_0 - I^2R$ in the last column. The last columns in Methods I. and II. each represent the iron losses and should be alike.

In Tables I., II., and III. and also Figs. 2, 3 and 4 the constants of the phasemeter as determined with the Kelvin balances were as follows:

$$I = hA = .814A$$

$$E = kB = 3.38B$$

$$W = IE \cos \varphi = \frac{hk}{4}(C_1^2 - C_2^2) = l(C_1^2 - C_2^2)$$

$$W = .687(C_1^2 - C_2^2).$$

Ring B had a mean diameter of 17.7 cm. and sectional area of 10.17 sq. cm. It was built up of 50 steel plates each having a thickness of .055 cm. Ring C had about half the mass of iron. Number 34 soft Swedish iron wire was used in the construction. The wires were carefully insulated with asphaltum.

TABLE I. SEE FIG. 2.

Ring C of fine iron wire with constant induced E.M.F. $E' = 33.8$. Induction constant at about 11,300 lines per sq. cm. Distributed magnetizing and exploring coils.

$I = .814A$ $\quad E = 3.38B$ $\quad W = .687(C_1^2 - C_2^2)$

Air-gap	A^2	I	I^2R	Method II.				Method I.				
				$(B')^2$	$C_1^2 - C_2^2$	Cos ϕ'	$W' = H$	B_0^2	$C_1^2 - C_2^2$	Cos ϕ_0	W_0	$W_0 - I^2R$
.00	4	1.63	1.85	100	122–86	.450	24.7	103	126–91	.430	24.1	22.2
.25	7	2.15	3.3	100	124–88	.340	24.7	103	130–92	.355	26.1	22.8
1	19	3.54	9.04	100	138–101	.212	25.2	104	143–99	.248	30.2	21.2
3	53	5.92	25.8	100	171–135	.124	24.7	106	195–124	.236	48.8	23.0
5	83	7.41	41.2	100	200–164	.099	24.7	108	233–145	.239	60.5	19.3
9	130	9.3	65.5	100	246–211	.077	24.1	108	295–170	.264	93.	27.5

PLATE I

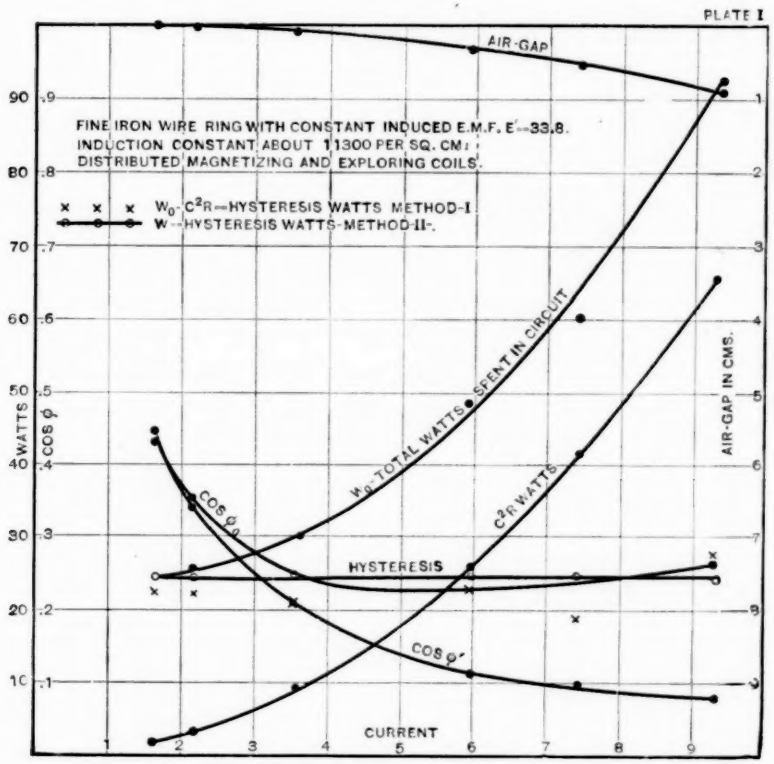

Fig. 2.

TABLE II. SEE FIG. 3.

Ring C of fine iron wire with constant induced E.M.F. $E' = 33.8$. Induction constant at about 11,300 lines per sq. cm. Parallel magnetizing and exploring coils.

$I = .814A$ $\quad E = 3.38B$ $\quad W = .687(C_1^2 - C_2^2)$

Air-gap	A^2	I	I^2R	Method II.				Method I.				
				$(B')^2$	$C_1^2-C_2^2$	Cos ϕ'	$W'=H$	B_0^2	$C_1^2-C_2^2$	Cos ϕ_0	W_0	W_0-I^2R
.0	5	1.82	2.65	100	119-88	.346	21.3	103	122-91	.347	21.3	19.7
1.0	13	2.93	7.05	100	129-100	.201	20.0	104	135-98	.252	25.4	18.4
3.0	28	4.3	15.5	100	140-115	.118	17.2	107	155-108	.215	32.2	16.7
5.0	45	5.45	25.	100	156-133	.086	15.8	105	175-119	.204	38.4	13.4
9.0	60	6.3	33.7	100	170-150	.064	13.7	104	198-128	.222	48.	14.3
19.0	79	7.25	45.	100	187-170	.047	11.7	106	226-140	.235	59.	14.0
Inf.	101	8.2	59.5	100	208-192	.04	11.0	106	253-153	.252	72.	12.5

PLATE II

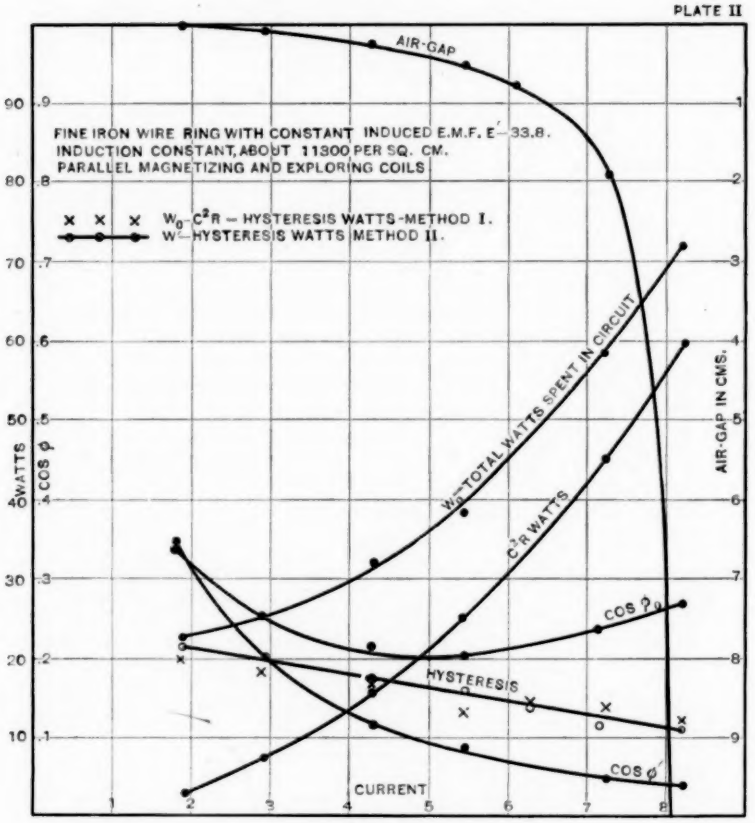

Fig. 3.

TABLE III. SEE FIG. 4.

Ring B of laminated steel with constant induced E.M.F. $E' = 33.8$. Induction constant and equal to 6,200 lines per sq. cm. Distributed magnetizing and exploring coils.

$I = .814A$ $\quad E = 3.38B$ $\quad W = .687(C_1^2 - C_2^2)$

Air-gap.	A^2	I	I^2R	Method II.				Method I.				
				$(B')^2$	$C_1^2-C_2^2$	Cos ϕ'	$W'=H$	B_0^2	$C_1^2-C_2^2$	Cos ϕ_0	W_0	W_0-I^2R
.00	.5	.57	.252	100	112–90	.78	15.1	103	114–92	.767	15.1	14.85
.15	1.8	1.09	.915	100	114–90	.446	16.5	104	116–93	.420	15.8	14.89
.40	3.5	1.52	1.8	100	115–91	.320	16.5	103	118–93	.329	17.2	15.4
1	10	2.57	5.2	100	121–98	.184	15.8	104	126–99	.209	18.5	13.3
3	35	4.81	18.5	100	147–124	.098	15.8	104	163–117	.191	31.6	13.1
5	59	6.25	31.6	100	171–148	.075	15.8	105	196–134	.197	42.6	11.0
9	96	7.97	52.7	100	207–185	.056	15.1	106	246–158	.218	60.5	7.8
19	159	10.2	91.	100	270–247	.045	15.8	110	342–198	.272	99.0	8.0
Inf.	322	14.6	196.	100	431–407	.033	16.5					

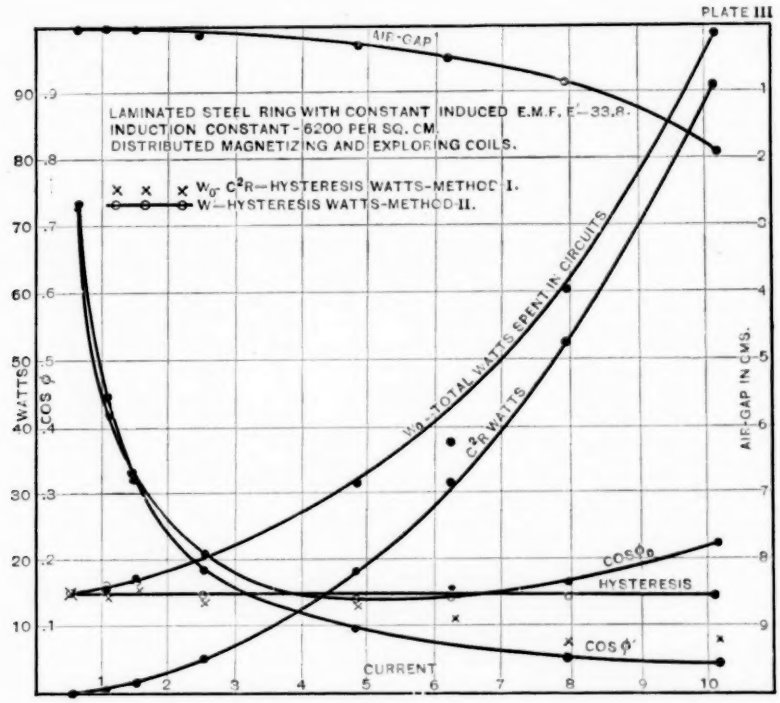

Fig. 4.

In general a very rapid fall in the power-factor, $\cos \varphi_0$, of the circuit was produced by a slight air-gap. This is best illustrated in Fig. 4. The joints in the laminated ring were ground to a fit while those in the fine iron ring were very imperfect. In Fig. 4 $\cos \varphi_0$ fell from .7 to .2 for an increase in the air-gap from zero to 1 mm. For larger air-gaps there was, however, an increase in $\cos \varphi_0$.

For small air-gaps $\cos \varphi'$ closely follows $\cos \varphi_0$. The curve for $\cos \varphi'$ may be considered as the limiting position for $\cos \varphi_0$ as the resistance of the magnetizing coil is indefinitely diminished.

The relations between the various quantities are illustrated in Fig. 5. The induction in the iron is represented by the vector $O\varphi$. As the air-gap is opened the current OI increases in magnitude and approaches $O\varphi$ in phase. In fact α the "angle of hysteretic advance of phase," that is, the angle by which the sine curve representing the induction lags behind the one representing the current, is meas-

ured by the complement of φ'. Thus a, as shown by the plates continually diminishes. For small air-gaps E_0 and E' are each practically in quadrature with the induction. But OE_r is proportional to the current and thus causes the curve for cos φ_0 to rise for large air-gaps.

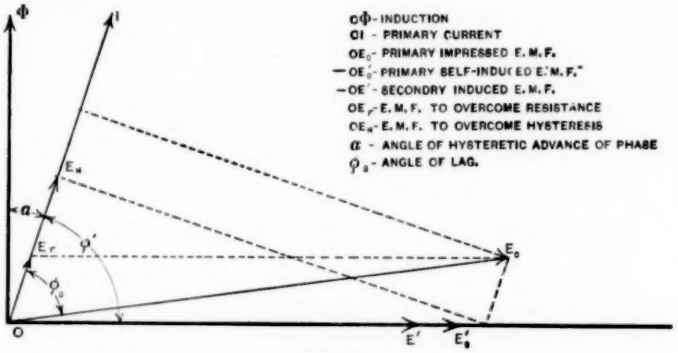

Fig. 5.

Providing there is no magnetic leakage of the magnetizing coil upon itself the components of OE_0 would be OE_0' and OE_r. The total watts spent in the circuit would be

$$W_0 = IE_0 \cos \varphi_0$$

and the hysteresis watts by Method I. would be given as

$$H = W_0 - I^2R = IE_0' \cos \varphi'.$$

But by Method II.

$$H = IE' \cos \varphi'.$$

Evidently the necessary condition that the hysteresis watts be the same by both methods is that there be no magnetic leakage.

It is evident from the tables that the values of the hysteresis watts by Method II. are much more consistent than by Method I. In order to obtain good results by Method. I extreme care was necessary in determining the current constant, h, the resistance of the magnetizing coil and its temperature. This was particularly true for large air-gaps since then the I^2R loss was so large as compared with the hysteresis watts.

It seems peculiar that the hysteresis watts should remain so nearly constant for large air-gaps. Fleming[1] has shown that straight iron strips in a uniform field have the loss of energy by hysteresis measured by the induction at about 22 per cent. of the length from the end. His curves show that the induction varies greatly over the length of the strip.

Self induction in the E.M.F. coil.—In all of the above it has been assumed that the current in the E.M.F. coil was in phase with the E.M.F. impressed at the terminals.

The effect of self induction in the pressure coil is to diminish the amplitude of the current in it to cos ε times its value were there no self induction; and to make the apparent angle of lag less than the true angle.

The diminution in the amplitude in the present case would be less than .02 of 1 per cent. Hence the ratio of the apparent watts to the true watts would be the ratio of cos φ as measured with the phasemeter to cos $(\varphi + \varepsilon)$. The percentage error would thus be about

$$\frac{\cos \varphi - \cos (\varphi + \varepsilon)}{\cos (\varphi + \varepsilon)}.$$

The calculated value of the coefficient of self induction of the coil S was

$$L = .0057 \text{ henry.}$$

Assuming the frequency to be 60 cycles per second and 500 ohms in series with S

$$\tan \varepsilon = .00185 \quad \text{or} \quad \varepsilon = 6'.$$

Taking cos $\varphi = .0031$, the lowest value given in the tables,

$$\frac{\cos \varphi - \cos (\varphi + \varepsilon)}{\cos \varphi} = \frac{.0331 - .0314}{.0331} = \frac{17}{331} \text{ or about 5 per cent.}$$

Taking in the second place cos $\varphi = .05$ the error would be 2.8 per cent.

[1] Phil. Mag., Sept., 1897, p. 262.

Greater sensibility with the phasemeter was obtained by increasing the number of turns in the current coil, M. The range, however, for different power-factors was diminished.

In Table IV. are given some results under the same condition as in Table I. except that the current coil had 12 turns instead of 3. The induction in this table was also slightly smaller, $E' = 32.5$ instead of $E' = 33.8$. The mean value of the hysteresis watts was 22.7 against 24.7 in Table I. Assuming the hysteresis to be proportional to the 1.6th's power of the induction the loss in the second place should have been 23 instead of 22.7.

It should be noticed that in Table IV. the hysteresis watts correspond to a difference of 142 in C_1^2 and C_2^2 against 36 in Table I.

TABLE IV.

Ring C of fine iron wire with constant induced E.M.F. $E' = 32.5$ volts. Induction constant at about 10,900 lines per sq. cm. Distributed magnetizing and exploring coils.

$I = .1975\ A$ $\quad E = 3.25\ B$ $\quad W = .1605\ (C_1^2 - C_2^2)$

Air-gap.	A	I	I^2R	Method II.				Method I.				
				B^2	$C_1^2-C_2^2$	Cos ϕ'	$W'=H$	B^2	$C_1^2-C_2^2$	Cos ϕ_0	W_0	W_0-I^2R
.00	69	1.64	1.86	100	238-97	.425	22.6	101	242-97	.437	23.5	22.65
.10	90	1.87	2.50	100	261-119	.375	22.8	106	269-118	.396	24.2	21.70
.25	116	2.13	3.26	100	286-145	.327	22.6	101	295-140	.352	24.9	21.60
.50	179	2.64	5.13	100	374-205	.266	22.8	101	364-201	.303	26.2	21.10
1.	277	3.29	8.05	100	447-307	.210	22.5	101	465-290	.262	28.1	20.05

The hysteresis loss of energy was determined for the fine wire ring with the ballistic galvanometer. For a maximum flux of 68,600 lines the hysteresis for 60 cycles per second was 31.4 watts. With the galvanometer calibrated with the standard cell and condenser instead of the current inductor it was 32 watts.

The maximum flux corresponding to an induction of 11,300 lines in the tables was 63,500. For this flux, assuming the hysteresis proportional to the 1.6th's power of the flux, the hysteresis for 60 cycles per second should have been 27.8 watts. The mean value given by the phasemeter method, Table I., was 24.7.

Thus the static and phasemeter methods do not give consistent values for the hysteresis. This may be due to the fact that the static measurements were very rough or that the law of the 1.6th power does not hold for such high inductions, or still further, that the frequency may not have been 60.

I should like to take this opportunity to thank Dr. Gray for many valuable suggestions and also for the interest which he has taken in the work throughout.

ROSE POLYTECHNIC INSTITUTE,
TERRE HAUTE, INDIANA.

ON THE MAGNETIC EFFECT OF ELECTRICAL CONVECTION. II.

By Harold Pender, Ph.D.

SINCE the publication of the results of his first experiments on the magnetic effect of a moving charged body,[1] the author has continued his investigations, with results in every way confirmatory of those of the previous experiments. A brief account of these later experiments may not be without interest, as they were performed under entirely new and more favorable conditions, and gave results which are far more consistent than those previously obtained.

M. Cremieu,[2] in criticising the previous paper of the author, suggested that the agreement between the observed and calculated values of the magnetic effect of the moving charged discs was due to the fact that the speeds and potential of the discs were of such critical values that a slight leak in the insulation would produce the observed effect. The first step then was to test this criticism by varying the speeds and potential within as great limits as possible. To do this, the same method as that previously used by the author, and first introduced by M. Cremieu, was adopted; namely, to measure the current induced in a coil when the charge on a rapidly rotating disc close to it is suddenly reversed.

The great difficulty encountered in the experiments of last year was the impossibility of shielding the needle of the sensitive galvanometer employed to detect this current, from the disturbing magnetic effects of the electric circuits in the vicinity of the laboratory, although the experiments were conducted at night after the electric cars had ceased running. Through the kindness of Professor Ames, Director of the Physical Laboratory of the Johns Hopkins University, I was enabled to move the entire apparatus to the country. The apparatus was, therefore, set up at McDonogh School, twelve

[1] Phil. Mag., 2, p. 169, 1901; Phys. Rev., 13, p. 203, 1901.

[2] Jour. de Phys., Dec., 1901.

miles from the city of Baltimore and two miles from the nearest electric car line. Experiment showed that this car line was too distant to affect the galvanometer. Electric power for running the various motors was furnished by the school.

The room first put at my disposal was a large garret. The apparatus was set up here, and considerable time was spent in vain attempts to mount the galvanometer so as to be free from mechanical jarring. This was finally given up as impossible, as it was found that the wind caused the whole building to rock considerably. The apparatus was then taken down and set up once more in a large basement room of another building. This room was 14.5 x 19.5 meters in size and had a cement floor, which was so solid that when the galvanometer was mounted on a stout table the needle was entirely free from mechanical jarring.

The various parts of the apparatus and their general arrangement were essentially the same as employed last year. There were, however, a few changes made, which will be briefly noticed.

The disc apparatus was changed only in a minor point, which, however, obviated a source of great inconvenience. This change was the mounting of the brushes making contact with the surfaces of the discs in such a way that the insulation of the cores could be cleaned without taking the apparatus apart. Special care was taken to clean this insulation thoroughly before each set of readings.

A new needle for the galvanometer was made, having a greater sensibility than the old one, and the ground-glass scale was placed two meters from the galvanometer. With the new needle and scale thus arranged it was possible to get four times the sensibility previously secured. But it was found more advantageous to sacrifice sensibility to steadiness of the needle, so that the galvanometer was usually adjusted to have only about one and a half times the sensibility of last year, so that a current of $.7 \times 10^{-10}$ ampères gave a deflection of 1 mm. on a scale 2 meters distant. At this sensibility the spot of light was ideally steady, the zero position seldom varying more than 2 mm. during the time required for the determination of a deflection (the period of the needle was about 35 seconds), although there was sometimes a slow drift of the light to the left. Any error due to this effect was easily eliminated by taking a

reading first on one side of the zero position and then on the other.

On the shaft of the combined reverser and commutator which served to reverse the sign of the charge on the discs and to commutate the galvanometer terminals, a second reverser was mounted and connected in series with a Daniell's cell, resistance boxes and the test coil *T*, on the disc apparatus. My idea was to adjust the value of the current through this coil until its effect on the large coil *I*, between the discs, was just equal and opposite to the effect of the moving charged discs, thus employing a zero method. It was found more convenient, however, to note first the deflection produced by passing a known current through the test coil with the discs discharged, then to break the circuit of the conduction current, connect in the Voss machine so as to charge the discs and note the deflection produced by the convection current; then again note the effect of the conduction current, and so on. This was accomplished by means of a set of switches operated by the observer at the galvanometer. The conduction current was adjusted so as to give about the same deflection as the convection current. Its value was determined from the known resistance of the circuit and the E.M.F. of the Daniell cell. The zero method was abandoned on account of the great length of period of the galvanometer needle, nearly a minute being required to detect any slight variation from the zero position.

The above method of procedure obviated the necessity of determining the sensibility of the galvanometer and the speed of the reverser for each set of readings and also the constant "A" of the former paper. Much trouble was experienced in getting brushes for the conduction current reverser which would make a steady contact, but finally this difficulty was overcome by making the brushes of very soft copper foil, each brush consisting of ten layers of foil. It was necessary in the course of the experiments to replace these brushes several times.

As before, a Voss machine and battery of Leyden jars were used to charge the discs. Instead, however, of connecting each pole of the Voss machine to the inside coats of three jars, one pole of the Voss was earthed and the other connected to the inside coating of

all six jars; the outer coats of which were earthed. The employment of this method rendered possible the developing of a higher potential, but the convection current produced by the moving charged discs was no longer reversed, but simply made and broken by the reverser. The reverser was run at such a speed that this make and break occurred about ten times a second.

The potential to which the discs was charged was measured by a Thomson electrostatic voltmeter having a range of 0–1,200 volts. This instrument was carefully calibrated by comparison with the standard guard ring electrometer at the Physical Laboratory of the Johns Hopkins University.

The following diagram, Fig. 1, will make clear the general arrangement of the apparatus.

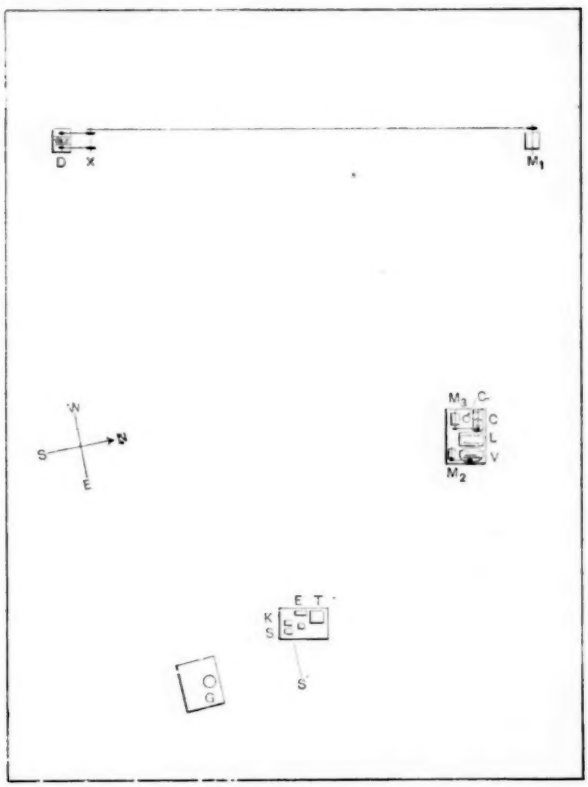

Fig. 1.
Scale 1 cm. = 2 m.

The letters indicate the following:

D the discs and induced coil.
M_1 motor driving the discs.
X countershaft.
C commutator and reverser.
M_3 motor driving the same.
L Leyden jars.
V Voss machine.
M_2 motor driving the same.
E electrometer.
T resistance boxes and switch in test circuit.
S high potential switch.
K key in galvanometer circuit.
G galvanometer.
S' scale for same.

Connections are omitted for the sake of clearness. They were essentially the same as previously used, except that a separate circuit had to be employed to carry the test current through the new reverser.

A number of sets of readings was now taken to determine the relation between the deflection d of the galvanometer produced by rapidly reversing a known current in the test coil, and commutating the galvanometer terminals connected with the induced coil, and the deflection D, produced by charging and discharging the discs at the same rate, the galvanometer terminals being commutated as before. The deflections actually measured were $2d$ and $2D$ (see preceding paper). In each set of readings five determinations of both $2d$ and $2D$ were made with the discs running in each direction. $2d$ and $2D$ were determined alternately. In the first five sets of readings the potential was kept practically constant and the speed of the discs varied from 9.9 to 92.4 revolutions a second, and in the next six sets the speed was kept practically constant and the potential of the discs varied from 905 to 5,900 volts. To show how closely the various quantities could be determined, the following table, giving all the data for one complete set of observations, is appended. A comparison of this table with Table III. of the previous paper will show how greatly the conditions of the experiment have been

improved. In the latter the greatest deviation from the mean of $2D$ is 42 per cent., while in the table below it is only 4.4 per cent.

The symbols have the following meanings: E is the electrometer reading, from which is deduced V, the potential of the discs in volts, by means of the calibration curve of the electrometer; T_1, the number of seconds required by the east disc to make 1,990 revolutions; T_2, the same for the disc on the west side; N the mean number of revolutions of the two disc per second deduced therefrom; i the test current measured in ampères; d and D as described above. The ratio of the two systems of units is calculated from these data in the manner described below.

TABLE I.

March 25, 1902. *Direction of Rotation: east disc* +, *west disc* −.

$2d$	$2D$	E	T_1	T_2	i
73	78	59.7	37.8	42.4	$.389 \times 10^{-4}$
75	81	61.0	38.8	42.2	
72	80	59.5	38.6	42.4	
73	82	60.5	38.8	42.0	
72	85	61.0	38.2	41.8	
73.0	81.2	60.3	38.4	42.2	$.389 \times 10^{-4}$

$V = 6310$ *volts.* $N = 49.3$. *Hence* $v = 3.00 \times 10^{10}$.

Direction of Rotation: east disc −, *west disc* +.

72	83	60.5	38.6	42.2	$.389 \times 10^{-4}$
72	79	61.0	39.0	42.6	
73	85	59.8	38.6	42.8	
72	80	58.0	38.0	42.4	
75	81	59.5	38.4	42.2	
72.8	81.6	59.8	38.5	42.5	$.389 \times 10^{-4}$

$V = 6260$ *volts.* $N = 49.1$. *Hence* $v = 2.95 \times 10^{10}$.

Mean value of v: 2.98×10^{10}.

To compare the observed values of $2D$ with those which should be expected on the assumption that the magnetic effect of a moving charged body is similar to that produced by a conduction current, the ratio V of the two systems of electric units was calculated in the same manner as described in the previous paper. No attempt was made at an accurate comparison until all the observations had been completed.

Referring to my previous paper, we have

$$v = \frac{4VN\mathcal{J}}{(B-\beta)D}\left[\mu + \nu\frac{B}{\pi}\log_e\left(2\cos\frac{\pi\beta}{2B}\right)\right]$$

where the symbols have the meanings there given. (I take this opportunity to call attention to two misprints in my former paper, namely, the symbol π for V in the formula on pp. 224 and 225, and $4v$ for $8V$ in the formula on p. 222.) In deducing this formula, the assumption was made that the charge on the discs was reversed from a positive to an equal negative value and vice versa by the reverser. In all the experiments here described the discs were alternately charged and discharged, sometimes negatively and sometimes positively. Hence to apply to these experiments, the above formula must be written

$$v = \frac{2VN\mathcal{J}}{(B-\beta)D}\left[\mu + \nu\frac{B}{\pi}\log_e\left(2\cos\frac{\pi\beta}{2B}\right)\right].$$

$\mathcal{J}$ is the deflection produced by reversing unit current in the test coil at the same rate as the discs are charged and discharged. Hence if d is the deflection produced by the current i under the same conditions,

$$\mathcal{J} = \frac{d}{i}.$$

If the potential is measured in volts and the current in ampères, the above formula then becomes

$$v = \frac{2VNd}{30(B-\beta)Di}\left[\mu + \nu\frac{B}{\pi}\log_e\left(2\cos\frac{\pi\beta}{2B}\right)\right].$$

μ and ν were determined in the manner described in the former paper. The distance between the condensing plates was kept constant. While the discs were running at a high speed one day in the early spring, the east disc flew off the axle, and was so badly damaged that it could not be used again. Instead of waiting to have it repaired, which would have taken considerable time, I went ahead with the remaining disc, making a virtue of necessity by thus varying the conditions of the experiment.

The constants in the above formula are as follows :

Mean Value for the Two Discs.	Value for the West Disc.
$B = 2.432$	$B = 2.469$
$\beta = .356$	$\beta = .356$
$\mu = 115.0$	$\mu = 115.0$
$\nu = 28.8$	$\nu = 28.8$

It so happened that the two discs were arranged almost perfectly symmetrically with respect to the coil, so that μ and ν for the two sides were identical within the limits of accuracy of measurement. Hence for the two discs

$$v = 4.18 \frac{VNd}{iD}$$

and for the single disc

$$v = 2.06 \frac{VNd}{iD}.$$

Below are given the mean values, determined from a series of observations similar to that recorded in Table I., of the variable quantities in the above formulæ for the various speeds and potentials employed. Those sets in which the single disc was used are marked with a *.

TABLE II.

i	$2d$	$2D$	N	V	v
$.103 \times 10^{-4}$	20.5	15.5	9.94	5690	3.03×10^{10}
.229	38.8	36.8	25.6	6250	3.08
.389	72.9	81.5	49.2	6275	2.98
.577	92.6	83.9	63.0	5960	3.00
.792	113.4	115.8	92.4	6230	2.98
From former paper.		67.2	102.2	3110	3.00
.0813	20.5	18.8	59.4	905	3.01
.134	26.0	32.7	59.6	2030	3.00
*.1005	18.7	22.0	59.2	2950	3.04
*.249	83.5	56.1	57.6	4090	2.92
*.249	47.1	38.1	57.9	5010	2.97
*.273	54.0	47.1	58.4	5900	2.98
				Mean	3.00

Surely no more conclusive refutation of M. Cremieu's criticism could be desired than that contained in the above results. Such close agreement between the observed and calculated effect under such varying conditions can not be ascribed to the effect of a con-

duction current caused by any leak in the insulation, for such a current would certainly, from its very nature, be independent of the speed. It is to be noted, however, that the exact coincidence of the value of v as determined from the above experiments with its known value must be considered as an accident only, for an error of at least one per cent. might readily have occurred in the determination of the constants of the apparatus.

In one of his experiments on the magnetic effect of a rotating charged disc on a magnetic needle suspended near it,[1] M. Cremieu observed a deflection of the needle when between it and the charged disc there was only a single condensing plate connected to earth, but when between the needle and this plate a second metallic plate connected to earth was introduced, no effect could be obtained. It therefore seemed worth while to try a similar experiment with the apparatus above described. In this experiment only one disc was used. Between the condensing plate next the coil and the coil itself was introduced a brass plate 1.5 mm. thick connected to earth. On setting the disc in rotation, but without charging it, a great unsteadiness of the needle was noticed. It was discovered that this was due to traces of iron in the brass plate. By gently tapping the plate when the disc was at rest the same effect could be produced. To keep the plate sufficiently steady to make any observations on its shielding effect, it was therefore necessary to run the disc at a very low speed. With the disc running at such a low speed, the Voss machine was connected in, and the deflection of the galvanometer needle observed. Then, without making any other change, the brass plate, connected to earth, was introduced, and the deflection again noted. This was done several times. The mean of a number of readings with the plate out and in were respectfully 12.8 and 13.0 mm. deflection. From this we can conclude that the introduction of the plate was without any such effect as noted by M. Cremieu. It may be of interest to note in this connection that an attempt to use a solid aluminum disc in place of the gilded disc in my first experiments was foiled by the magnetic disturbances caused by traces of iron in the aluminum, although the purest metal obtainable was employed.

[1] C. R., 131, p. 797, 1900.

The next experiment tried was the application of Cremieu's method to the investigation of the magnetic effect of a dielectric moving in a uniform electrostatic field, or in other words, the magnetic action of a moving apparent charge. This question was first investigated by Roentgen in 1888.[1] Roentgen showed that a moving polarized disc was capable of deflecting a magnetic needle suspended near it. The maximum deflection observed by Roentgen was 3 mm. One objection offered to Roentgen's experiment was that the effect observed might have been caused by the disc assuming a real charge by leakage across from the condensing plates on each side of it. Cremieu's method precludes any such action as this, inasmuch as the condensing plates are rapidly charged and discharged, and even though there should be a slight leakage of a real charge across to the surface of the disc, this charge could produce no deflection of the galvanometer, as the charge would tend to assume a constant value, whereas the deflection of the galvanometer is due to a *change* in the electrical condition of the disc.

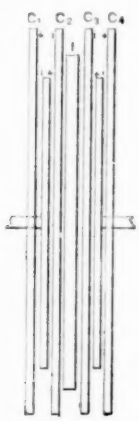

Fig. 2.
½ actual size.

For this experiment two ebonite discs were made, diameter 30.5 cm., thickness .8 cm. The gilded micanite discs used in the first experiment were replaced by these (see Fig. 2). The two condensing plates C_2 and C_3 next to the induced coil I were earthed. The other two condensing plates C_1 and C_4 were connected through the interrupter to the pole of the Voss machine so that they might be rapidly charged and discharged. Everything else remained exactly the same as in the first experiments. When the two outer plates C_1 and C_4 are charged positively, for example, the two ebonite discs become polarized so that the surfaces next to the outer condensing plates assume an apparent negative charge, and the surfaces nearer the inner condensing plates C_2 and C_3 assume an apparent positive charge. Since the positively charged surfaces are nearer the induced coil than the surfaces negatively charged, when the outer plates are suddenly discharged while the discs are rotating, there will be a slight current induced in the coil I. Since the two

[1] Wied. Ann., 40, p. 93.

surfaces oppose each other in their magnetic action, the resultant effect is very small, being greater the thicker the discs, for the same surface density of the apparent charge. A slight deflection of the galvanometer was observed with the discs arranged as just described, but it was found that a greater effect could be obtained by mounting the two discs flat up against each other on the same axle, thus using only one side of the apparatus. This arrangement amounted to the use of a single disc 1.60 cm. thick, *i. e.*, twice the thickness of one of the discs. Also, it was possible to charge to a higher potential the single condenser thus formed than the two condensers above described, for a given speed of the Voss machine and the interrupter. (It may be here noted that the Voss machine was always run at the highest possible speed. A machine of greater capacity would have made possible the procuring of a greater deflection.)

To calculate the deflection which should be expected on the assumption that a moving apparent charge has a magnetic action, a method similar to that employed in the previous calculation was adopted. Let σ be the surface density of the apparent charge on the surface of the disc next to the coil, assumed uniform as a first approximation, N the number of revolutions of the disc per second, v the ratio of the two systems of magnetic units, r the radius of an imaginary ring on the surface of the disc with its center at the center of the disc and of width dr, δ the deflection of the galvanometer needle produced by a unit current in such a ring on the surface of the disc next to the induced coil, rapidly made and broken the same number of times a second as the disc is polarized and depolarized, δ' the same for a unit current in such a ring on the opposite surface of the disc. Then the deflection of the galvanometer needle due to the rapid polarizing and depolarizing of the rotating disc will be

$$D = \frac{2\pi N\sigma}{v}\int_0^R r(\delta - \delta')dr.$$

σ is determined from the formula (Webster, Elec. and Mag., p. 364)

$$\sigma = \frac{(\mu - 1)V}{4\pi[\mu(d_1 + d_2) + d]},$$

where μ is the dielectric constant of the disc (for ebonite 2.5), d_1

the distance between the outer condensing plate C_1 and the surface of the disc, d_2 the distance between the inner condensing plate C_2 and the surface of the disc, d the thickness of the disc, and V the potential of the outer plate, the inner plate being earthed.

The integral

$$\int_0^R r(\partial - \partial')dr$$

was determined by a "calibration of the apparatus" in a manner similar to that employed in the first experiments. A set of coils of known radii was clamped up against the surface of the disc next to the induced coil. A known current i was sent through the reverser and one of these coils. While the current was being thus reversed in this coil, the frame carrying the disc was drawn back from the induced coil a distance equal to the thickness of the disc, and the resultant change A in the galvanometer deflection noted. The frame was then pushed up into its former position, and the change in deflection again noted. A second known current i_1 was then sent through the reverser and the test coil on the frame carrying the disc, and the change in galvanometer deflection B resulting from a known change $(i_1 - i_2)$ in this current noted. i, i_1, and i_2 were so chosen that the deflection produced by the current i_1 in the test coil was equal to the deflection produced by the current i in the coil on the surface of the disc, and the change in deflection A was approximately equal to the change in deflection B. In this way the quantities A and B were measured at the same part of the galvanometer scale, thus avoiding any error due to a lack of proportion between the current and deflection, which was considerable in the galvanometer employed. Let ρ_1 be the ratio of the deflection produced by unit current flowing through the reverser and any coil on the surface of the disc next to the induced coil to the deflection produced by unit current flowing through the test coil, ρ_2 the corresponding quantity when the coil is on the opposite surface of the disc. Then

$$\rho_1 - \rho_2 = \frac{A}{i} \times \frac{(i_1 - i_2)}{B}.$$

From the observations taken as above described $\rho_1 - \rho_2$ was calculated and plotted for twelve different coils. Let J be the deflection

produced by a unit current flowing through the reverser and test coil. Then

$$\delta - \delta' = \tfrac{1}{2} \Delta(\rho_1 - \rho_2)$$

($\delta - \delta'$ is the deflection due to the making and breaking of a current, whereas A and B are the deflections resulting from a reversal of current, hence the factor $\tfrac{1}{2}$). The formula for D therefore becomes

$$D = \frac{\pi N \Delta \sigma}{v} \int_0^R r(\rho_1 - \rho_2) dr.$$

The integral

$$\int_0^R r(\rho_1 - \rho_2) dr$$

was calculated graphically from the plot of $\rho_1 - \rho_2$.

A number of observations of the deflection D were made, which always agreed in direction and fairly well in amount with the deflection as calculated. A close agreement could not be expected, inasmuch as the assumption that σ is uniform at the edge of the disc is only a rough approximation at the truth, and even under the best conditions, the deflection is necessarily small. The following data, which are the mean values from one set of readings, will suffice to illustrate the capabilities of the method.

$d = 1.60$, $d_1 = .30$, $d_2 = .75$.

$V = 24.9$ C.G.S. electrostatic units ($= 7{,}470$ volts).

$\sigma = .70$ electrostatic units.

$\Delta = 2.51 \times 10^7$ mm. per electromagnetic unit.

$N = 57.8$

$\int_0^R r(\rho_1 - \rho_2) dr = 22.8.$

$2D$ calculated 4.85.

$2D$ observed 4.5.

$2D$ was the actual deflection measured, not D.

Considering the importance of the question as to the magnetic action of a moving static charge, the following experiment, in which was observed the direct magnetic action of a moving charged disc on a magnetic needle suspended near it, may be of interest, though similar results have been obtained by other experimenters. The uninjured one of the two micanite discs used in the first experiments was provided with a row of sixteen brass studs set at equal intervals

apart in a circle of five centimeters radius around the center of the disc. The gilded surface of the disc was then divided into sixteen sectors on each side, each pair of sectors carrying a stud. The sectors were separated from one another by a strip of micanite surface 1 centimeter in width. The tin foil on both the condensing plates, with the exception of a sector on each twice the width of a sector on the disc, was removed. The tinfoil sectors were earthed, and the sectors on the disc could be connected one at a time,

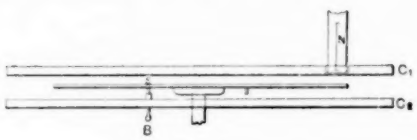

Fig. 3.

through a brush B (Fig. 3) set so as to make contact with the studs S, with one pole of the Voss machine, the other pole of which was earthed. The frame carrying the disc was so arranged that the disc could be set in rotation about a vertical axis. A hole about 2 cm. in diameter was cut through the upper ebonite plate C_1 diametrically opposite the tin foil segment, and so that its center came 1 cm. over the edge of the disc. This hole was covered on the side next to the disc with a thin sheet of mica. Fitting loosely into the hole so as not to touch the sides or the mica plate at the bottom, was a brass tube in which was suspended a delicate astatic needle N. The two magnets forming the needle were 5 cm. apart. The needle and attached mirror weighed about 3 mg., and was suspended by a fine quartz fiber. With the control magnet properly placed the needle could be given a period of 25 seconds. The case in which the needle was suspended was fixed to a frame built over the disc apparatus, having an independent support, so that when the disc was set in rotation there was no jarring of the needle. The position of the needle was read by the reflected image of an electric light filiament on a ground-glass scale two meters distant.

When the disc was set in rotation and the brush making contact with the studs was connected to the Voss machine, a deflection of the needle was obtained, which was in the proper direction and of the proper amount to be accounted for on the assumption of the

magnetic action of a moving charge. The arrangement here adopted precludes any conduction of charge in the condensing plates or in the disc itself, two possibilities which have been suggested to account for the deflection observed with solid condensing plates and a disc of uniform conducting metallic surface. The results of one set of observations will suffice to give an idea of the magnitude of the quantities involved.

Mean distance between the two surfaces of the disc and needle, 1.61 cm.

Distance between the two condensing segments (one on each ebonite plate C_1 and C_2), 2.18 cm.

Thickness of disc, .356 cm.

Potential of the disc, 5,000 volts.

Speed of disc, 69.7 revolutions per second.

Observed deflection, 47.9.

Calculated deflection, 56.0.

The calculation was made in a manner similar to that employed in the previous experiments, the principle of it being a calibration of the disc apparatus such as above described. Only a rough calculation was attempted, as it would be a matter of some difficulty to calculate the exact distribution of the charge on the sectors. As a first approximation this distribution was assumed uniform. The agreement between the observed and calculated values of the deflection is therefore as good as could be expected.

My chief object in setting up the apparatus with the magnetic needle was to test experimentally certain criticisms made in my former paper on M. Cremieu's experiments, especially his experiments on open electric circuits.[1] However, due to unavoidable delays, it was not until the first of June that I was ready to proceed with the work. But now the damp and sultry weather of the summer had set in and put an end to all experiments with static electricity for the time being.

I wish here to express my thanks to Professor Ames, of the Johns Hopkins University, to whose kindness I am indebted for the opportunity for carrying out these experiments, and whose criticisms and suggestions have proved of great value.

BALTIMORE, MD., July 1, 1902.

[1] C. R., 132, p. 1108.

THE TEMPERATURES AND SPECTRAL ENERGY CURVES OF LUMINOUS FLAMES.

BY G. W. STEWART.

IN a communication to the German Physical Society, Messrs. Lummer and Pringsheim[1] suggested a method for the measurement of luminous flame temperatures, based upon the well-known relation between the wave-length of the maximum in the energy curve and the temperature of a black body. This relation is $\lambda_m T = A$, a constant, where λ_m is the wave-length of the maximum and T is the absolute temperature. For a black body they found the constant to be 2940, while experiments with bright platinum gave about 2630. They suggested that the temperatures of luminous flames could be calculated from this formula, and that the temperature would lie between the values obtained by assuming these two values for the constant.

In a recent article[2] the writer computed the temperature of the acetylene flame according to this suggestion. The value obtained for a cylindrical flame lay between 2890° C. and 2560° C., which this is certainly wrong. I have since found my method of correcting the observed curves to be incorrect, but the temperatures, using the correct curves, are yet much too high.

Kurlbaum,[3] in an account of a simple method for the measurement of flame temperatures, remarks that while a luminous flame of very great thickness would radiate as a black body, yet we might expect a candle flame, for example, to radiate as a very thin film of carbon at that temperature. He has already shown[4] that a thin coat of lampblack is more transparent to long than to short waves, *i. e.*, the absorption and radiation increase with decrease of wave-

[1] Lummer and Pringsheim, Verh. Deutsch. Phys. Gesell., 1899, p. 214.
[2] PHYSICAL REVIEW, XIII., 1901, p. 257.
[3] Kurlbaum, Physik. Zeitschr., Feb. 1, 1902, p. 187.
[4] Kurlbaum, Annal., 67, 1899, p. 846.

length. Hence in the curves showing the distribution of energy in the spectrum of the acetylene flame, for example, the maximum would be found further toward the short wave-lengths than in the case of a black body at the same temperature. One would expect, then, that the results obtained by applying the formula, $\lambda_m T = A$, would give temperatures much too high.

Kurlbaum's method of measuring flame temperatures is as follows: Let a black body at a known temperature be brought before the eye of the observer. It will appear of a certain brightness. If a flame is now inserted between the eye and the black body, the apparent brightness of the body will, in general, change. If the flame has the higher temperature, the body will appear brighter, and if it has a lower temperature, the brightness will be less. But if the body and the flame have the same temperature, the brightness of the body will remain unchanged. One then needs only to vary the temperature of the body until the brightness remains the same when the flame is introduced. By this method, Kurlbaum found the temperature of the candle flame to be 1430° C.

Lummer and Pringsheim,[1] in a later article, criticise Kurlbaum's method. They show that it would be correct only in case the reflection of the flame can be neglected in comparison with the absorption. They made a measurement of the transmission of the candle flame in the red portion of the spectrum and found the value 0.8. Then assuming the reflection of the incandescent carbon particles in the flame to be from 5 to 10 per cent., they computed, by means of a formula and their own previous work on a black body, that Kurlbaum's value of 1430° C. or 1700° abs. is about 40° to 120° too low. When this correction is made, the result 1740° abs. to 1820° abs. compares favorably with their value of 1780° abs. to 1960° abs., found from the spectral energy curve. They conclude that the slight discrepancy hardly justifies the assertion that the incandescent carbon particles in the flame of the candle radiate more selectively than bright platinum. Kurlbaum, however, replies,[2] and shows by experiment that the reflection of the candle is about 1 per cent. in the red portion of the spectrum. If he now assumes the

[1] Lummer and Pringsheim, Phys. Zeitschr., 3, No. 11, 1902, p. 233.
[2] Kurlbaum, Phys. Zeitschr., 3, No. 15, p. 332.

value of transmission found by Lummer and Pringsheim, 80 per cent., and applies this correction to his previous value of the temperature, 1430° C., there appears to be an error of only 7°. He therefore concludes that the error to which his attention has been called, is of no importance.

The work which follows in the present paper shows that the value of Lummer and Pringsheim for the transmission of the candle flame is much too small; that, nevertheless, their criticism of Kurlbaum's method is of considerable importance; that Kurlbaum's statement to the effect that the carbon particles in a luminous flame radiate more selectively than bright platinum, is correct; and that the value of Lummer and Pringsheim for the wave-length of the maximum in the energy curve for the candle flame is in error.

It is a well-known fact that the horizontal distribution of light of an ordinary illuminating gas flame varies but a few per cent., showing that the flame is practically transparent for visible radiations. According to Kurlbaum,[1] the transparency in the infra-red is even greater. Although his observations were made with carbon at ordinary temperatures, the shifting of the maximum absorption toward the longer waves with increasing temperature, as shown by Koenigsberger,[2] would probably not be sufficient to make the transparency in the visible greater than that in the infra-red.

We know, too, from Kirchoff's Law, — the ratio of the emissive to the absorbing power for any wave-length of a body at any temperature is equal to the emissive power of a black body at the same temperature and wave-length — that the maximum absorption for carbon at the temperature of the acetylene flame is at least less than 1.05 μ; for this wave-length, at which I have found (see below) the energy of the acetylene flame to be a maximum, should lie between the wave-length of the maximum absorption of the flame and the wave-length of the maximum emission of a black body at that temperature. And, assuming as correct the temperature determined by Nichols,[3] 1900° C., this latter wave-length is about 1.35 μ.

From these two facts, viz., that the maximum absorption of these luminous particles is probably not beyond the visible and certainly

[1] Kurlbaum, Annal., 67, 1899, p. 846.
[2] Koenigsberger, Annal., 4, 1901, p. 796.
[3] Nichols, PHYS. REV., X., 1900, p. 234.

at a much less wave-length than 1.05 μ, and that an ordinary gas flame absorbs very little visible radiation, we can conclude that the absorption of any wave-length (the ultra-violet excepted) by the luminous particles of a flame is quite small. This would indicate that the movement of the maximum in the spectral energy curves of different luminous flames would depend almost entirely upon the temperature, for the amount of carbon present in any ordinary flame is too small to affect appreciably this maximum by its selective absorption. If this is true, the formula, $\lambda_m T = A$, should also apply to luminous flames, provided the constant A is determined for some flame of which the temperature is known.

Before taking up this application of the formula, I shall give an account of some experiments undertaken to give a more definite knowledge of the absorption of the acetylene flame.

The Absorption of the Acetylene Flame.

At one end of a photometer bar about 6 meters in length, I mounted two burners, giving cylindrical flames 3.5 cm. in height and about 3.5 mm. in maximum diameter, upon a support which could be revolved about an axis midway between the two flames. These burners were 3 cm. apart. At the other end of the bar was placed a flat jet Naphey burner, its flame being about equal in intensity to the sum of the other two. The readings were made with a Bunsen photometer. By revolving the upright holding the two burners, the intensity of the two cylindrical flames, as compared with the flat flame, was observed in three cases: First, one cylindrical flame behind the other; second, the two side by side; third, the reverse of the first. The adjustments had to be made with great care as a very slight error in setting would produce a large error in the result. Representing the intensity of the light, relative to that of the flat flame, by I, and indicating the observation, as numbered above, by the subscript, the percentage absorption of such a cylindrical flame would be

$$\frac{I_2 - \frac{I_1 + I_3}{2}}{\frac{I_2}{2}}.$$

The results obtained from one series of observations were 4.5 per cent., 2.4 per cent., 2.7 per cent., and 4.7 per cent., and the average 3.6 per cent. Two days later with a new adjustment, and by computing after averaging all the readings obtained, the result was again 3.6 per cent. Several errors enter into this determination, but all tend to increase the apparent absorption. There is a reflection of the light of one flame from the carbon particles and also from the hot gases in the other, and the latter flame also acts as a diverging lens. It is sufficient for our purpose to know that the actual absorption does not exceed, at the most, 3 or 4 per cent.

As the acetylene flame is quite rich in carbon, the small percentage here obtained, viewed in the light of the remarks made above, assures one that the formula, $\lambda_m T = A$, should apply to flames in which the source of radiation is incandescent carbon. Of course there are present emission bands due to H_2O and CO_2. But, fortunately, all these bands are at wave-lengths greater than 1.4 μ, and hence have no influence on the position of the maximum in the energy curves, excepting with flames whose temperature is considerably lower than the gas flame, *e. g.*, the candle flame.

THE APPLICATION OF $\lambda_m T = A$ TO LUMINOUS FLAMES.

In order to test the correctness of applying $\lambda_m T = A$ to luminous flames, I shall consider the energy curves for the candle and illuminating gas flames, finding the value of A from previous work on the acetylene flame. These energy curves were found with the same apparatus, mirror spectrometer, radiometer and fluorite prism, as in the case of the acetylene flame.[1] The adjustment was tested by means of the CO_2 band at 4.4 μ, and no error was found in the measurement of wave-lengths.

The Acetylene Flame. — In Fig. 1, a portion of the observed curve of a cylindrical acetylene flame is reproduced from my previous article. Curve 1 is the observed curve, and Curve 2, the correct one, is obtained by applying Paschen's equation,

$$af(x) = F(x) - \tfrac{1}{6} F_1(x),$$

where a is the slit width of spectrometer and radiometer in wave-lengths.

[1] PHYSICAL REVIEW, XIII., 1901, p. 257.

$f(x)$ is the intensity of the wave-length x on the corrected curve.

$F(x)$ is the intensity of the wave-length x on the uncorrected curve.

and $$F_2(x) = \frac{F(x+a) - F(x-a)}{2} - F(x).$$

In a later article I hope to give experimental justification of the correctness of Paschen's method as opposed to the one I used in my previous article.

The maximum point on the corrected curve is 1.05 μ.

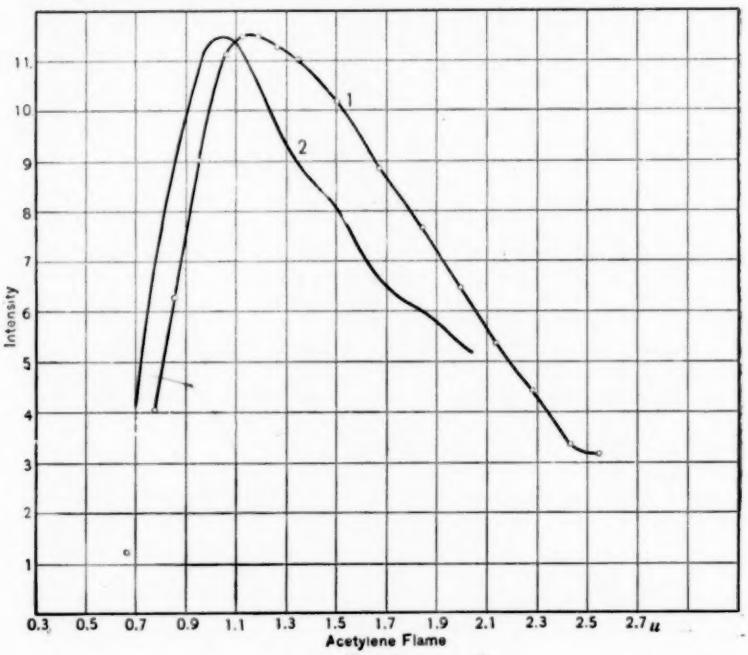

Fig. 1.

The Gas Flame. — In Fig. 2, Curve 2, is shown the spectral energy curve of a cylindrical flame of illuminating gas. Curve 1 is the observed distribution of energy. That the observations were made with a fair degree of accuracy, is readily seen by the appearance, in proper proportion, of the emission bands of H_2O at 1.46 μ and 1.90 μ in the corrected curve.

The maximum value of the intensity occurs at 1.13 μ.

The Candle Flame. — The energy curve of the candle flame is

very difficult to obtain with accuracy on account of the great variation in the intensity of the flame itself and the change of its proper

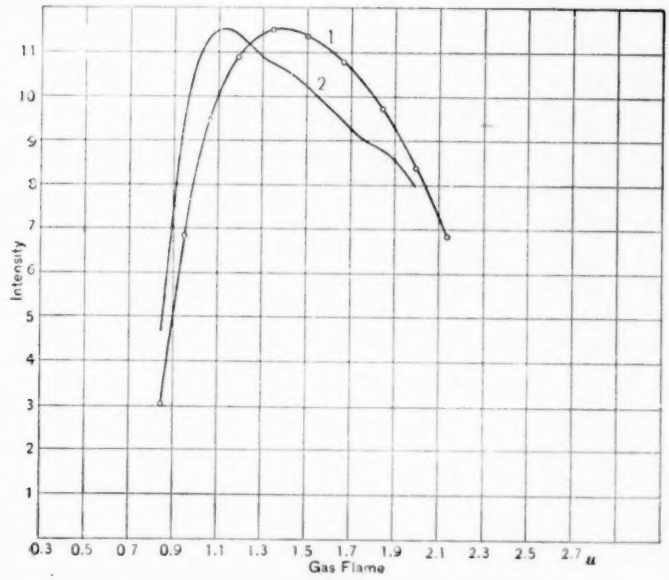

Fig. 2.

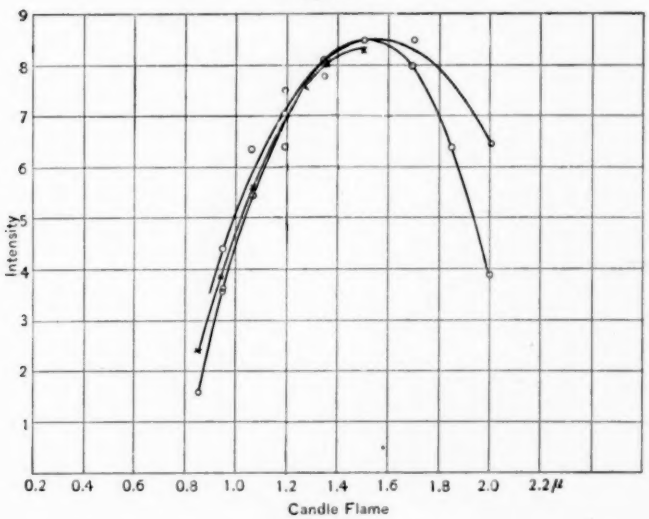

Fig. 3.

position in front of the spectrometer slit. In Fig. 3 are shown three observed curves. When these curves are corrected, the aver-

age maximum is about 1.4 μ, and the corrected curves clearly show that the position of this maximum is influenced by the H_2O band at 1.46 μ. If one makes allowance for this band, the average maximum of curves due to the incandescent carbon alone, appears to be about 1.25 μ. So far as I am aware, the only previous work on the energy curve of the candle flame is that of Lummer and Pringsheim in the articles cited at the beginning of this paper. They give no observations whatever, merely stating that the maximum occurs at 1.5 μ. This is about the value found on my uncorrected curves.

Temperatures Computed from $\lambda_m T = A$.—In order to find the value of A in the case of luminous flames, I assume the temperature of the acetylene flame to be 1900° C., found by Nichols in an article already cited, and the wave-length of the maximum of its energy curve to be 1.05 μ. A becomes 1.05×2170, or about 2282.

Computing T from the wave-lengths of the maxima for the gas and candle flames, 1.13 μ and 1.25 μ respectively, I find that with the former T is 2020° abs., or about 1750° C., and with the latter, T is 1825° abs., or about 1550° C. In the paper already mentioned, Nichols finds the temperature at the center of the luminous sheath of a flat gas flame to be 1780° C., and the temperature of the surface of the luminous cone of a candle flame to vary from about 1400° C. to 1675° C., depending upon the height from the rim of the cup of the candle. Judging from his results, the average temperature of the part I had in front of the spectrometer slit is about 1630° C.

The error in the value of 1430° C. found by Kurlbaum is due to his error in neglecting the reflection in comparison with the absorption. He found the reflection to be about 1 per cent., but he makes a mistake in assuming the value of the transmission of the candle flame found by Lummer and Pringsheim, 80 per cent., to be correct. My value of transmission of the acetylene is 96.4 per cent. In comparing the two flames in regard to absorption, it must be remembered that the acetylene flame is very rich in carbon, and that also the luminous portion of the candle flame does not average more than twice the thickness of the acetylene flame I used. In consideration of these facts, it seems quite probable that the transmission of the candle flame is about 95 to 96 per cent. Then,

too, the error due to the absorbing flame acting as a diverging lens, makes the apparent transmission too small. If this should make an error of 1 per cent., which is not unlikely, and if the reflection of the flame is 1 per cent., the transmission would be 96 or 97 per cent., or the actual absorption of the flame would be 2 or 3 per cent. If this estimate is approximately correct, then Kurlbaum's error is as great as stated by Lummer and Pringsheim, *i. e.*, his value is 40° to 120° too low. Even this correction does not satisfactorily account for the disagreement between the values of Nichols and Kurlbaum.

In comparing my results for the temperature obtained from the energy curves with those Nichols obtained by actual measurement, several points must be considered. He measured the temperatures by means of thermo-junctions of varying cross-section, plotted a curve with temperatures as ordinates and relative cross-sections as abscissæ, and by extrapolation found the temperature corresponding to a junction of zero cross-section. Although he does not assume a great degree of accuracy for these values obtained by extrapolation, yet the method should give fairly consistent results for different flames. I find, however, that from his results on the gas flame one can draw a different curve, apparently just as good, which gives a value of 1760° C., or 20° lower.

His temperature of 1900° C. for the acetylene flame is for the center of the luminous sheath of a flat flame, while that of 1780° C. for the illuminating gas and 1630° C. (averaged value) for the candle, are of the edge of the luminous sheath. If he had measured the temperature of the center of the luminous sheath of the gas flame, he would have found it perhaps 15° or 20° lower.

These facts indicate that, if the maxima of my curves were exactly correct, and if the formula $\lambda_m T = A$ applies to luminous flame I should get a value of the gas flame lower than 1780° C. by perhaps 20° or 30°, and for the candle flame, a value lower than 1630° C.

My value of 1750° C. for the gas flame is therefore in very good agreement with the value given by Nichols, which that of the candle flame, 1550° C. is as nearly correct as the accuracy of my observations on the distribution of the energy in its spectrum would lead one to expect.

Summary.—Lummer and Pringsheim criticise Kurlbaum's method of measuring the temperatures of flames in that it neglects the reflection in comparison with the absorption of the flames. This criticism is just.

Kurlbaum's assertion that the carbon particles in a luminous flame radiate more selectively than bright platinum, is correct.

The maxima in the energy curves of luminous flames depend almost entirely upon the temperatures of the flames, and hence the law for a black body, $\lambda_m T = A$, should be applicable to flames. The value of A is determined experimentally in the case of the acetylene flame, and found to be 2282. Then equation $\lambda_m T = 2282$ is applied to the candle and gas flames, and the values for temperatures found to agree satisfactorily with the observed valves of Nichols.

PHYSICAL LABORATORY, CORNELL UNIVERSITY,
July, 1902.

AN ALTERNATING CURRENT MODEL.

BY W. P. WHITE.

IN view of the fact that models illustrating the mechanical analogies of alternating current phenomena, though often described, are not commonly used, it has seemed worth while to give an account of such a model, which has proved very useful in the University of Wisconsin, and which can be made in half a day by almost any one, at practically no cost.

In this machine, that which corresponds to the driving electromotive force is furnished by the hand of the person operating it; mechanical friction gives an imperfect but adequate imitation of resistance, while inertia and the tension of an elastic cord correspond to self induction and the action of dielectrics where there is capacity. In order to show the action on the current of each of these four components of the electromotive force, the current itself is represented by the motion of a body separate from all of them and as free as possible from the action of any force, frictional or other. This condition is secured by using a light, balanced lever, oscillating on two sharp points. This, the "current lever," is shown in Fig. 2 at *C*, and is of light wood, about 2 x 2 x 100 cm. It is attached by means of a vertical piece of wood to the crossbar, *Y*, from each end of which a large, sharpened nail, *N*, projects downward, resting on the uprights, *U*, which are simply pieces of wood nailed on opposite sides of a box, *B*.

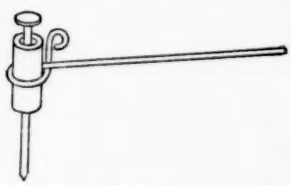

Fig. 1.

The action upon the current lever, *C*, of the various bodies whose inertia, elasticity, etc., represent the components of the electromotive force is made much easier to understand and follow by means of the "phase indicators," shown at 1, 2, 3, 4, Fig. 2, and alone in Fig. 1. Each one is a very small tube, turning on a nail, with a wire

soldered to it so as to form a bent lever, with arms 1 cm. and 8 cm. long. The nail is driven into the current lever, and the long arm of the phase indicator is allowed a few centimeters' play between two other nails; the acting body—that is, the body representing resistance or self-induction, etc.—instead of being fastened directly to the current lever, is connected by a projecting pin to the loop in the short arm of the phase indicator. Thus, according as the body is urging the current one way or the other, the indicator will point to the right or left, and whenever the direction of pull changes the indicator will fly from one stop to the other with a sharp click, which emphasizes strongly the beginning of a new phase, and which enables all four sets of phase relations to be followed and studied while the machine is in motion.

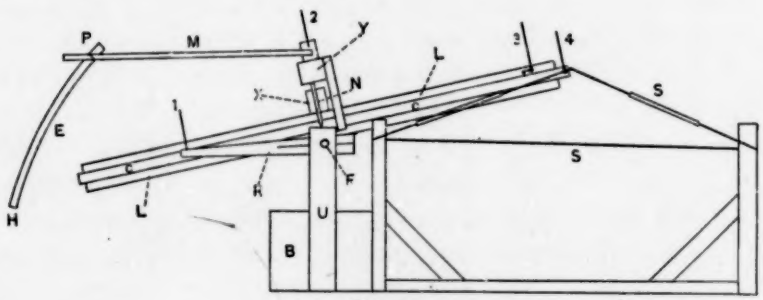

Fig. 2.

The driving force is transmitted longitudinally along the stiff rod, *M*, to the phase indicator, 2. This ought to be an undulating pressure, rising and falling without too great suddenness. It is more difficult to give such a pressure with the hand alone than might be supposed, on account of a tendency the mind has, in directing the action, to will the motion and not the pressure. Now as the machine is most useful in the hands of the students themselves it ought not to require any special practice or skill on the part of the operator. Accordingly, the flexible rod, *E*, is tied at *P* to *M*; when *E* is bent to and fro by the hand holding it at *H* it gives a pressure which depends on the amount of bending, and is quite satisfactory.

The effect of self induction is given by the lever, *L*, which swings on two nails driven through its cross bar, *X*, inside of and in line

with the nails, *N*. It is heavily loaded at the ends to give high moment of inertia, is attached to the current lever, *C*, when desired, by the phase indicator at 3, and is tipped down to the left when not in use. From four to eight kg. is a suitable weight; this lever, like the current lever, should be carefully adjusted so as to be in neutral equilibrium.

The elastic loop, *S*, is stretched if the current lever is displaced either way from a middle position. The reaction due to this mechanical displacement imitates the action of the dielectric of a condenser, which opposes the current with a force proportional to the electric displacement. The loop is made of string and stout rubber bands. The analogue of resistance is the bar, *R*, turning about the metal rod, *F*, as the lever turns, with friction which can be varied by the screw which clamps *R* on *F*.

If the lever, *L*, and the loop, *S*, are disconnected, and the machine operated, we have a representation of a simple circuit without self-induction or capacity. The two phase indicators, 1 and 2, keep time with each other, as the current lever swings with the impulses of the driving force. If capacity is now added to the make-believe circuit, by simply stretching the loop, *S*, over its frame three distinct clicks are heard every oscillation from the three indicators, 1, 2 and 4, which are now in different phases. The force exerted on the current lever by the resistance is still in step with it; the capacity phase indicator flies over as the current-lever passes the middle point, and is half an oscillation ahead; the interval between the clicks of phase indicators 1 and 2 marks the lead, whose dependence on varied resistance, capacity, and frequency may be studied as fully as desired.

In class demonstration the machine readily shows the motion of the current, lag or lead, and the phase relations of the different forces, as affected by variations in frequency, resistance and capacity. It is especially useful in making clear the reason for certain phenomena which are puzzling to the beginner. For instance, it shows clearly how a capacity, which resists the current, can at the same time cause it to "lead," how self induction and capacity can neutralize each other, and how resonance depends on the frequency of the electromotive force.

NEW BOOKS.

Primary Batteries: Their Theory, Construction and Use. By W. R. Cooper, M A., B.Sc. London, Electrician Publishing Co.

Mr. Cooper's book, which has been promised for some time, is a welcome and useful addition to the literature of a subject which has been for for some time attracting much attention. A large amount of original work has been done in this field within the last decade. Whatever may be the ultimate merits of the ionization and osmotic pressure theories, there can be no question that they have been a great stimulus to investigations in the subject of voltaic reactions. Probably more has been added to our knowledge of the science of this subject during the past fifteen years than during the preceding fifty. So many investigations have been made that even the diligent student finds it difficult to keep up with all of them. They appear in the journals of physics, chemistry and physical chemistry, and the thorough investigator must search the scientific periodicals of two continents for his materials.

The entire subject of voltaic cells is still in a state of flux, and conclusions must be accepted with caution. Under such circumstances it is no easy task to write a connected account of what is known regarding primary cells, in such a way as to set down ascertained truth on the one hand, and avoid controverted conclusions on the other. Mr. Cooper must be credited with very fair success in this difficult field, to say the least. Nothing so good has appeared since the rise and progress of the dissociation theory.

A good and clear account is given of the modern theories on which Nernst's formula for the E.M.F. of a voltaic cell is based, and the same theory is applied to the elucidation of concentration cells. The chapter on Standard Cells contains a full account of the standards now in use A table of constants relating to Standard Cells brings together all the values of the E.M.F. and temperature coefficient obtained by different investigators.

In so good a book one hesitates to call attention to errors that diminish its value. To do so, however, is only to bring them to the author's attention before the preparation of a second edition.

It is to be regretted that he still insists on naming the zinc the positive plate, while at the same time he says that "the positive pole of a cell — viewing it, as it were, from the external circuit — is a part of what

is called the negative plate when looking at the cell from the point of view of its internal working." This old confusion should have eliminated itself long ago. In fact a long list of distinguished writers might be enumerated in which the zinc plate or its equivalent is invariably called the negative. So long as the attempt was made to name the peroxide plate of a storage cell the negative during the discharge and the positive during the charging stage intolerable confusion reigned. The only solution was to call it the positive plate or electrode all the time. The copper or the carbon is at best only a conductor for the reception and conveyance of the positive charges conveyed to it by the ions. What possible valid reason is there then for calling it the negative plate? I do not recall another recent writer who has not discarded the old nomenclature for the more modern and rational one, in which the zinc is negative, whether looked at from the point of view of the internal or the external circuit.

Attention is called to a statement which has done frequent duty in one class of German publications. The author has copied Nernst, as have most of Nernst's followers, with a notable exception or two. On page 43 he speaks of bringing a quantity of heat H from a temperature $T + dT$ down to a temperature T. Also on page 45, occurs the following: "The heat H really falls from the temperature $T + dT$ down to T." It is respectfully submitted that *heat* can not be brought up or down from one temperature to another. Heat is energy and has no temperature. The language of the old caloric theory is wholly out of place in a book employing modern thermodynamic reasoning. Lehfeldt in his "Physicial Chemistry" and Jones in his new book on the same subject have happily discarded this antique form of statement.

HENRY S. CARHART.